CM2021/12/01

Wednesday, December 1, 2021 10:00 AM

$$2\rho F \frac{\partial \bar{e}}{\partial C} F^t : D = T : D$$

to prove this?

Start with Balance of linear momentum

$$\boxed{\frac{D}{Dt} P = F_{surface} + f_{body}}$$

$$\frac{D}{Dt}\int_{B_t} \underbrace{\rho v}_{P} dV_y = \int_{\partial B_t} \underbrace{t_y}_{T \cdot n_y} dA_y + \int_{B_t} \underbrace{\rho b \, dV_y}_{dF_{body}}$$

$d F_{surface}$

$$\frac{D}{Dt}\int_{B_t} \vec{v} \underbrace{(\rho dV_y)}_{dm} = \int \frac{Dv}{Dt} (\rho dV_y)$$

reduced transport

$$\int_{B_t} \underbrace{\frac{Dv}{Dt}}_{\vec{a}} \rho \, dV_y = \int_{\partial B_t} T \cdot n_y \, dA_y + \int_{B_t} \rho b \, dV_y$$

$$\int_{B_t} \rho \frac{Dv}{Dt} dV_y = \int_{B_t} \operatorname{div} T \, dV_y + \int_{B_t} \rho b \, dV_y$$

$$\int_{B_t} \left(\rho \frac{Dv}{Dt} - \operatorname{div} T - \rho b\right) dV_y = 0$$

B_t arbitrary, use localization

$$\to \boxed{\rho \frac{Dv}{Dt} - \operatorname{div} T - \rho b = 0} \quad (1a)$$

linear momentum $\to$ balance energy (power) $(\times \vec{v})$

$1a \cdot \vec{v}$

$$\boxed{\rho \vec{v} \frac{D\vec{v}}{Dt} - \vec{v} \cdot \operatorname{div} T - \vec{v} \cdot \rho b = 0} \quad (1b)$$

$$T = 2\rho F \frac{\partial \bar{e}}{\partial C} F^t$$

$$= F \underbrace{\left(\frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}}{\partial C}\right)}_{G(C)} F^t$$

for hyperelastic material

$$\mathcal{C}_{ijkl} = 4\rho \frac{\partial^2 \bar{e}}{\partial C_{ij} \partial C_{kl}}$$

$\mathcal{C}_{ijkl} = \mathcal{C}_{klij}$ major sym.

Voigt stiffness

$\mathcal{C}_{6 \times 6}$ will become sym

36 independent val $\to$ 21

Balance of energy (First law of Thermodynamics)

$$\frac{D}{Dt} \underset{\text{energy}}{\mathcal{E}} = \int_{\partial B_t} \vec{v} \cdot d\vec{F}_s + \int_{B_t} \vec{v} \cdot \underbrace{\rho \vec{b}}_{d\vec{F}_v} + \cdots$$

other terms

$$\cancel{\int_{B_t} Q \, dV_y}$$

~~heat source~~

~~Joule's heating~~

No thermal & electro magnetic effect (Mechanical response only)

for hyperelastic material

$\bar{e}(C)$

No thermal & electromagnetic effect (Mechanical response only)

for hyperelastic material $\bar{e}(C)$

$$\frac{D}{Dt}\int_{B_t}\Big(\underbrace{\frac{\rho |v|^2}{2}}_{\text{kinetic energy density}} + \rho \underbrace{e}_{\text{internal energy density}}\Big)\,dV_y = \int_{\partial B_t}\vec{v}\cdot T n\, dA_y + \int_{B_t}\vec{v}\cdot\rho b\, dV_y$$

$$\frac{D}{Dt}\int_{B_t}\frac{\vec{v}\cdot\vec{v}}{2}\underbrace{(\rho\, dV_y)}_{dm} + \frac{D}{Dt}\int_{B_t}\bar{e}(C)\underbrace{(\rho\, dV_y)}_{dm} = \int_{\partial B_t}(vT)n\, dA_y + \int_{B_t}\vec{v}\cdot\rho\vec{b}\, dV_y$$

reduced transport $\quad \frac{D}{Dt}\int_{B_t} f\rho\, dV_y = \int_{B_t}\frac{Df}{Dt}\rho\, dV_y$

$$\int_{B_t}\Big(\frac{1}{2}\frac{D\, v\cdot v}{Dt} + \frac{D\bar{e}}{Dt}\Big)\rho\, dV_y = \int_{B_t}\mathrm{div}(vT)\, dV_y + \int_{B_t} v\rho b\, dV_y$$

Note $\quad \frac{1}{2}\frac{D\,\vec{v}\cdot\vec{v}}{Dt} = \frac{1}{2}\frac{D\vec{v}}{Dt}\cdot v + \frac{1}{2}\vec{v}\cdot\frac{Dv}{Dt} = v\cdot\underbrace{\frac{Dv}{Dt}}_{a}$

$$\int_{B_t}\Big[\rho v\cdot\frac{Dv}{Dt} + \rho\frac{D\bar{e}}{Dt} - \mathrm{div}(vT) - \vec{v}\cdot\rho\vec{b}\Big]\, dV_y = 0 \qquad B_t \text{ arbitrary}$$

localization

$\rightarrow$

$$\boxed{\rho\, v\cdot\frac{Dv}{Dt} + \rho\frac{D\bar{e}}{Dt} - \mathrm{div}(vT) - v\cdot\rho b = 0} \qquad (2)$$

a, b, c, d

$$\boxed{\rho\vec{v}\,\frac{D\vec{v}}{Dt} - \vec{v}\cdot\mathrm{div}\,T - \vec{v}\cdot\rho b = 0} \qquad (7b)$$

a, c', d

(2) - (7b)

$$\boxed{\rho\frac{D\bar{e}}{Dt} - \mathrm{div}(vT) + v\,\mathrm{div}\,T = 0} \qquad (3)$$

$$\mathrm{div}(vT) = (v_i T_{ij})_{,j} = \underbrace{\left(\frac{\partial v_i}{\partial y_j}\right)}_{\text{spatial gradient of } v} T_{ij} + v_i\left(\frac{\partial T_{ij}}{\partial y_j}\right)$$

der w.r.t y_j $\quad f_{j,j} = \frac{\partial f_j}{\partial y_j}$

$(\nabla\cdot T)_i$

$\mathrm{div}\,T$

$$\mathrm{div}\,vT = L_{ij}T_{ij} + v_i\nabla T_i = L : T + v\cdot\mathrm{div}\,T$$

plug this in eqn 3

$$\rho\frac{D\bar{e}}{Dt} - (L:T + \cancel{v\,\mathrm{div}\,T}) + \cancel{v\,\mathrm{div}\,T} = 0$$

$$\rho \frac{D\bar{e}}{Dt} - (L:T + v\,\cancel{\mathrm{div}\,T}) + \cancel{v}\,\mathrm{div}\,T = 0$$

$$\boxed{\rho \frac{D\bar{e}}{Dt} = L:T} \quad (4)$$

hyperelastic

$\bar{e}(F) \longrightarrow \bar{e}(C)$

from objectivity

$\bar{e}(C)$

(5)

$$\frac{D\bar{e}(C)}{Dt} = \frac{\partial \bar{e}(C)}{\partial C_{ij}} \frac{DC_{ij}}{Dt}\Big|_{x\text{-fixed}}$$

material time derivative

will be computed next

$\left(\frac{\partial \bar{e}(C)}{\partial C}\right)_{ij}$ will comment on this below

$$\frac{D\bar{e}(C)}{Dt} = \frac{\partial \bar{e}}{\partial C} : \frac{DC}{Dt}$$

(6)

$\frac{DC}{Dt}$ term $\quad \left(\frac{DC}{Dt}\right)_{ij} = \frac{DC_{ij}}{Dt} = ?$, $C = F^t F$

$$(7) \quad \boxed{\frac{DC}{Dt} = \frac{DF^t}{Dt} F + F^t \frac{DF}{Dt} = \left(\frac{DF}{Dt}\right)^t F + F^t \left(\frac{DF}{Dt}\right)}$$

$$\left(\frac{DF}{Dt}\right)_{ij} = \frac{D\overbrace{\left(\frac{\partial y_i}{\partial x_j}\right)}^{F_{ij}}}{Dt} = \frac{\partial^2 y_i}{\partial t\, \partial x_j}\Big|_{x\text{-fixed}} = \frac{\partial^2 y_i}{\partial x_j\, \partial t}\Big|_{x\text{-fixed}}$$

material time rate

$$= \frac{\partial}{\partial x_j}\Big(\underbrace{\frac{\partial y_i}{\partial t}\Big|_{x\text{-fixed}}}_{v_i}\Big) = \frac{\partial v_i}{\partial x_j}$$

we want to express this with $\partial/\partial y$ terms

$$= \underbrace{\left(\frac{\partial v_i}{\partial y_k}\right)}_{L_{ik}} \underbrace{\left(\frac{\partial y_k}{\partial x_j}\right)}_{F_{kj}} \longrightarrow \left(\frac{DF}{Dt}\right)_{ij} = L_{ik} F_{kj} \longrightarrow \boxed{\frac{DF}{Dt} = LF}$$

$$\frac{DC}{Dt} = (LF)^t F + F^t (LF) = F^t L^t F + F^t L F = 2F^t \left(\frac{L + L^t}{2}\right) F$$

$D = \frac{L + L^t}{2}$ (sym L) stretching tensor

$W = \frac{L - L^t}{2}$ (skew L) spin tensor

(don't confuse D with stretch tensor U: $F = RU$)

$$\boxed{\frac{DC}{Dt} = 2F^t D F} \quad (8)$$

$$\frac{D\bar{e}(C)}{Dt} = \frac{\partial \bar{e}}{\partial C} : \frac{DC}{Dt} \quad \text{eqn } (6)$$

$$\frac{D\bar{e}(C)}{Dt} = \frac{\partial \bar{e}}{\partial C} : 2F^t D F$$

$$\frac{D\bar{e}(C)}{Dt} = \frac{\partial \bar{e}}{\partial C} : 2F^t DF$$

$$\rho \frac{D\bar{e}}{Dt} = L : T \text{ eqn } 4$$

$$\longrightarrow \boxed{2\rho \frac{\partial \bar{e}}{\partial C} : F^t DF = T : L} \quad (9)$$

$$(\qquad) : D \qquad T : D$$

modifying eqn 9

$$T : L = T : \underbrace{\text{sym} L}_{D} + T : \underbrace{\text{skew} L}_{W} = T : D + \underbrace{T : W}$$

$$\underbrace{T_{ij}}_{\text{sym}} \underbrace{W_{ij}}_{\text{skew}} = 0 \quad (HW1)$$

$T_{ij} = T_{ji}$
from balance of angular momentum

$$2\rho \frac{\partial \bar{e}}{\partial C} : F^t DF = 2\rho \left(F \frac{\partial \bar{e}}{\partial C} F^t \right) : D$$

HW assignment
Hint : use indicial notation

plug in eqn 9

$$\boxed{\underbrace{\left(2\rho F \frac{\partial \bar{e}}{\partial C} F^t\right)}_{\text{sym (?)}} : \underbrace{D}_{\text{sym}} = T : \underbrace{D}_{\text{sym}}} \quad (10)$$

$$\left(2\rho F \frac{\partial \bar{e}}{\partial C} F^t\right)^t = 2\rho (F^t)^t \left(\frac{\partial \bar{e}}{\partial C}\right)^t F^t = 2F \underbrace{\left(\frac{\partial \bar{e}}{\partial C}\right)^t}F^t$$

will discuss how $\frac{\partial \bar{e}}{\partial C}$ is calculated and why it's sym.

$$\overbrace{\left(2\rho \underbrace{F \frac{\partial \bar{e}}{\partial C} F^t}_{\text{sym}} - T\right)}^{=0} : \underbrace{D}_{\substack{\text{sym} \\ \text{arbitrary}}} = 0$$

if $A : B = 0$ for all sym B & A is sym then $A = 0$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} : \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{12} & B_{23} & B_{33} \end{bmatrix} = A_{11}B_{11} + A_{22}B_{22} + A_{33}B_{33} + 2A_{12}B_{12} + 2A_{13}B_{13} + 2A_{23}B_{23} = 0 \quad (*)$$

choose $B = 0$ except $B_{11} = 1$ $\longrightarrow$ $A_{11} = 0$

chose $B_{12} = 1 \rightarrow A_{12} = 0$ similarly $A_{22} = 0$, $A_{33} = 0$, $A_{13} = 0$, $A_{23} = 0$

$\rightarrow A = 0$

$$T = 2\rho F \frac{\partial \bar{e}(C)}{\partial C} F^t$$

$$= F \left(2\rho \frac{\partial \bar{e}}{\partial C} \right) F^t$$

$$= F \left(\frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}(C)}{\partial C} \right) F^t = F \bar{G}(C) F^t$$

Hyperelastic material $\quad \bar{G}(C) = \frac{2\rho_0}{\sqrt{\det C}} \frac{\partial \bar{e}(C)}{\partial C}$

$$\mathcal{C} = 2 \frac{\partial \bar{G}}{\partial C}\Big|_{C=1} \rightarrow \mathcal{C} = 4\rho_0 \frac{\partial^2 \bar{e}(C)}{\partial C \, \partial C}$$

$$\mathcal{C}_{ijkl} = 4\rho_0 \frac{\partial^2 \bar{e}}{\partial C_{ij} \partial C_{kl}} = C_{klij}$$

Note 1

How is $\frac{\partial \bar{e}}{\partial C}$ computed?

$\bar{e}(C) = C_{11} + C_{22} + C_{12} + 3C_{21} - 2$ ✓ $\qquad \bar{e}(I) = 0$

$\frac{\partial \bar{e}}{\partial C} = ? \qquad \left(\frac{\partial \bar{e}}{\partial C}\right)_{ij} := \frac{\partial \bar{e}}{\partial C_{ij}}$

$$\frac{\partial \bar{e}}{\partial C_{11}} = 1 \quad \frac{\partial \bar{e}}{\partial C_{22}} = 1 \quad \frac{\partial \bar{e}}{\partial C_{12}} = 1 \quad \frac{\partial \bar{e}}{\partial C_{21}} = 3 \rightarrow \frac{\partial \bar{e}}{\partial C} = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we know that $C_{21} = C_{12}$

should be written in terms of its independent components

☹

components

$$\bar e(C) = \bar e(\underbrace{C_{11}, C_{22}, C_{33}, \bar C_{12}, \bar C_{23}, \bar C_{31}})$$

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{31} \\ & C_{22} & C_{23} \\ \text{sym} & & C_{33} \end{bmatrix}$$

$$\frac{\partial \bar e}{\partial C_{12}}$$

$$\frac{\partial \bar e}{\partial C_{21}} = \cancel{\frac{\partial \bar e}{\partial C_{11}} \frac{\partial C_{11}}{\partial C_{21}}}^{\,0} + \cancel{\cdots} + \frac{\partial \bar e}{\partial C_{12}} \frac{\partial \bar C_{12}}{\partial C_{21}}^{\,1} + \cancel{\cdots}$$

$\frac{1}{2}$

$$\frac{\partial \bar e}{\partial C_{21}} = \frac{\partial \bar e}{\partial C_{12}} = \frac{1}{2} \frac{\partial e}{\partial \bar C_{12}} \qquad \frac{\partial \bar e}{\partial C} \text{ is sym.}$$

$$\bar e(C) = C_{11} + C_{22} + \cancel{C_{12} + 3C_{21}} - 2 \quad \cdots$$

$4C_{12}$

write it in a sym. fashion $C_{12} = C_{21}$

$$\bar e(C) = C_{11} + C_{22} + 2C_{12} + 2C_{21} - 2$$

non physical

$$\frac{\partial e}{\partial C} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{C} = \frac{\partial^2 e}{\partial C \partial C}$$

Another note

In the class we got EOM in Eulerian & Lagrangian from work

HW: **Exercise 96** *Prove Theorem 168. Base your proof on the statement of local balance of linear momentum in the referential description,*

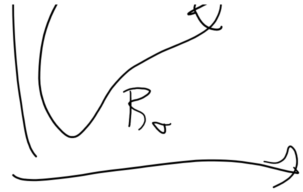
$$\nabla \cdot \mathbf{S} + \rho_0 \tilde{\mathbf{b}} = \rho_0 \mathbf{a} \text{ on } \overset{0}{\mathcal{B}} \times [t_0, \infty).$$

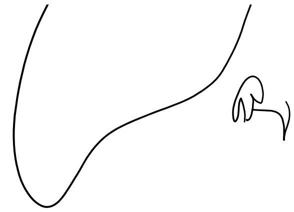
Calculate

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{P}} \frac{1}{2} |\mathbf{v}|^2 \rho_0 \, \mathrm{d}V_x,$$

$$\rho \int a v \, dV_u = \int t_u \, dA_u + \int \rho b \, dV_u$$

$$\frac{D}{Dt}\int_{B_t} \rho v \, dV_y = \int_{\partial B_t} \underbrace{t_y}_{T \cdot n_y} dA_y + \int_{B_t} \underbrace{\rho b \, dV_y}_{dF_{body}}$$





$$\frac{D}{Dt}\int_{B_0} \rho_0(X) v \, dV_x + \int_{\partial B_0} (\underbrace{T J F^{-t}}_{S}) \, d\vec{A}_x + \int \rho_0 b \, dV_x$$

$$d\vec{A}_y = J F^{-t} \, d\vec{A}_x$$

$$\int_{B_0} \left(\rho_0 \frac{Dv}{Dt} - Div(S) - \rho_0 b\right) dV_x = 0$$

localization

$$\boxed{\rho_0 \frac{Dv}{Dt} - Div(S) - \rho_0 b = 0}$$

$v \cdot$

to get sth like energy statement from this

$v \cdot$

$$\underbrace{\rho_0(X) v \cdot \frac{Dv}{Dt}}_{\frac{D \frac{1}{2} v \cdot v}{Dt}} - v \cdot Div(S) - \rho_0 v \cdot b = 0$$

$$\frac{D}{Dt}\left(\frac{1}{2}\rho_0 \, v \cdot v\right) - \underbrace{v \cdot Div(S)}_{\text{relate this to}} - \rho_0 v \cdot b = 0$$

$$\underbrace{Div(vS) \ \ldots \ \text{like above}}$$

$$D\rho_0 \frac{v \cdot v}{2} = \underbrace{v \cdot Div(S) + \rho_0 v b}$$

Calculate

$$\boxed{\frac{d}{dt}\int_{\mathcal{P}} \frac{1}{2}|\mathbf{v}|^2 \rho_0 \, dV_x,}$$

$$\frac{D}{Dt}\int_{P_0} \frac{1}{2}|v|^2 \rho_0 \, dV_x = \frac{D}{Dt}\int_{P}\left(\frac{1}{2}\rho_0(X) \, \vec{V}(x,t) \cdot \vec{v}(x,t)\right) dV_x$$

$$= \int \frac{D}{Dt}\left(\frac{1}{2}|v|^2 \rho_0\right) dV_x = \int \left(v \cdot Div S + \rho_0 v b\right) dV_y$$