Continuum Mechanics: Equation Sheet

Tensor Algebra

Indicial Notaiton

Kronecker's Delta

$$\delta_{ij} = \delta_{ji}$$
$$\delta_{ii} = 3 \qquad \delta_{ij}\delta_{ij} = 3$$
$$\delta_{ij} a_{p\cdots j\cdots q} = a_{p\cdots i\cdots q} \qquad \delta_{ij} a_j = a_i$$

Alternating symbol

Kronecker's Delta

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{jik} = -\epsilon_{kji} = -\epsilon_{ikj}$$
$$\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp} = \begin{vmatrix} \delta_{jp} & \delta_{jq} \\ \delta_{kp} & \delta_{kq} \end{vmatrix}$$
$$\epsilon_{ijk}\epsilon_{ijq} = 2\delta_{kq}$$
$$\epsilon_{ijk}\epsilon_{ijk} = 6$$

0.1 Determinant and Matrix Inverse

$$\det \mathbf{A} = \epsilon_{ijk} A_{1i} A_{2j} A_{3k} = \epsilon_{ijk} A_{i1} A_{j2} A_{k3}$$
$$\epsilon_{pqr} \det \mathbf{A} = \epsilon_{ijk} A_{ip} A_{jq} A_{kr} = \epsilon_{ijk} A_{pi} A_{qj} A_{rk}$$
$$\det \mathbf{A} = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr} A_{ip} A_{jq} A_{kr} = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr} A_{pi} A_{qj} A_{rk}$$
$$\epsilon_{ijk}\epsilon_{pqr} \det \mathbf{A} = \begin{vmatrix} A_{ip} & A_{iq} & A_{ir} \\ A_{jp} & A_{jq} & A_{jr} \\ A_{kp} & A_{kq} & A_{kr} \end{vmatrix}$$
$$A^{-1}_{rk} = \frac{1}{2\det A}\epsilon_{ijk}\epsilon_{pqr} A_{ip} A_{jq}$$
$$\mathbf{Ax} = \mathbf{b} \;\; (x_r = A^{-1}_{rk} b_k) \quad \Rightarrow x_r = \frac{1}{2\det A}\epsilon_{ijk}\epsilon_{pqr} A_{ip} A_{jq} b_k$$
$$\epsilon_{ijk} A_{im} A_{jn} = \epsilon_{mnp} \det(\mathbf{A}) \mathbf{A}^{-1}_{pk}$$
$$\frac{\mathrm{d}(\det \mathbf{A})}{\mathrm{d}\alpha} = \mathrm{trace}\left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\alpha}\mathbf{A}^{-1}\right) \det \mathbf{A} \quad \alpha \text{ any argument (dependency) of } \mathbf{A} \text{ such as time } t$$

Definition of Tensor Product

$$(\mathbf{a} \otimes \mathbf{b})\,\mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\,\mathbf{a}$$

Properties of the Tensor Product

$$(\mathbf{a}\otimes\mathbf{b})^T = (\mathbf{b}\otimes\mathbf{a})$$
$$(\mathbf{a}\otimes\mathbf{b})(\mathbf{c}\otimes\mathbf{d}) = (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}\otimes\mathbf{d}$$

Let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be an orthonormal set, then

$$(\mathbf{e}_i\otimes\mathbf{e}_i)(\mathbf{e}_j\otimes\mathbf{e}_j) = \begin{cases}\mathbf{0}, & \text{for } i\neq j,\\ \mathbf{e}_i\otimes\mathbf{e}_i & \text{for } i=j,\end{cases}$$
$$\sum_{i=1}^{3}\mathbf{e}_i\otimes\mathbf{e}_i = \mathbf{I}$$

Definition of Trace of a Tensor Product

$$\operatorname{tr}(\mathbf{a}\otimes\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}$$
$$\operatorname{tr}\mathbf{A} = \sum_{i=1}^{3}A_{ii}$$
$$\operatorname{tr}\mathbf{S}^T = \operatorname{tr}\mathbf{S}$$
$$\operatorname{tr}(\mathbf{S}\mathbf{T}) = \operatorname{tr}(\mathbf{T}\mathbf{S})$$
$$\mathbf{S}\cdot\mathbf{T} = \operatorname{tr}\left(\mathbf{S}^T\mathbf{T}\right)$$

Properties of the Inner Product

$$\mathbf{R}\cdot(\mathbf{S}\mathbf{T}) = (\mathbf{S}^T\mathbf{R})\cdot\mathbf{T} = (\mathbf{R}\mathbf{T}^T)\cdot\mathbf{S}$$
$$\mathbf{u}\cdot\mathbf{S}\mathbf{v} = \mathbf{S}\cdot(\mathbf{u}\otimes\mathbf{v})$$
$$(\mathbf{a}\otimes\mathbf{b})\cdot(\mathbf{u}\otimes\mathbf{v}) = (\mathbf{a}\cdot\mathbf{u})(\mathbf{b}\cdot\mathbf{v})$$
$$\mathbf{S}(\mathbf{a}\otimes\mathbf{b}) = (\mathbf{S}\mathbf{a})\otimes\mathbf{b}$$
$$(\mathbf{a}\otimes\mathbf{b})\,\mathbf{S} = \mathbf{a}\otimes\left(\mathbf{S}^T\mathbf{b}\right)$$
$$\sum_{i=1}^{3}(\mathbf{S}\mathbf{e}_i)\otimes\mathbf{e}_i = \mathbf{S}$$

Relation between Skew-symmetric Tensors and Cross-Product

There is a one-to-one correspondence between vectors and skew-symmetric tensors: given any skew-symmetric tensor $\mathbf{W}$ there exists a unique vector $\mathbf{w}$ such that

$$\mathbf{W}\mathbf{v} = \mathbf{w}\times\mathbf{v}, \quad \forall\mathbf{v}\in\mathcal{V}.$$

Furthermore, if $\{\alpha,\beta,\gamma\}$ are the components of the vector $\mathbf{w}$ with respect to an orthonormal basis, then

$$[W] = \operatorname{ax}(\mathbf{w}) = \begin{bmatrix}0 & -\gamma & \beta\\ \gamma & 0 & -\alpha\\ -\beta & \alpha & 0\end{bmatrix}.$$

Orthogonal tensors - Rotations

Any orthogonal tensor $\mathbf{Q}$ corresponds to a rotation (potentially with zero angle, *i.e.*, identity map), and a reflection if $\mathbf{Q}$ is improper, that is $\det \mathbf{Q} = -1$). If we have the axis $\mathbf{a}$ and rotation angle θ, we can express $\mathbf{Q}$ as,

$$\mathbf{Q} = \cos\theta \mathbf{I} + (1-\cos\theta)\mathbf{a}\otimes\mathbf{a} + \sin\theta \text{ax}(\mathbf{a})$$

Other useful identities are,

$$\mathbf{Q}\mathbf{a} = \pm 1\mathbf{a} \quad + \text{ for proper and - for improper orthogonal } \mathbf{Q}, \text{ that is } \mathbf{a} \text{ is an eigenvector of } \mathbf{Q}$$
$$\text{trace}(\mathbf{Q}) = 2\cos\theta + 1$$

Inversely, if we have the rotation $\mathbf{Q}$ and want to obtain the axis $\mathbf{a}$ and angle θ of rotation, we use the above equation to derive them as follows (see "Brannon_Rebecca_rotation.pdf" pages 43-44):

1. Obtain $c = \frac{\text{trace}(\mathbf{Q})-1}{2}$.
2. If $c = 1$, $\mathbf{Q} = \mathbf{I}$. That is, $\mathbf{Q}$ is an identity tensor, and any direction would be an axis of rotation with zero angle of rotation.
3. If $c = -1$, angle of rotation is $\theta = \pi(\text{radians}) = 180^\circ$. That is, $\mathbf{Q}$ is reflection with respect to the axis of $\mathbf{Q}$, $\mathbf{a}$. The axis of $\mathbf{Q}$ is obtained by normalizing any nonzero column of $\mathbf{Q}+\mathbf{I}$
4. If $|c| \neq 1$ we need to compute the angle and axis of $\mathbf{Q}$ as follows,
   (a) $s = +\sqrt{1-c^2}$.
   (b) Compute the axis of rotation:

$$\mathbf{a} = \text{ax}(\mathbf{Q}) = \frac{1}{\sin\theta}\text{ax}\left[\text{skew}(\mathbf{Q})\right] \quad \Rightarrow \quad \begin{aligned} a_1 &= \tfrac{1}{2s}(Q_{32}-Q_{23}) \\ a_2 &= \tfrac{1}{2s}(Q_{13}-Q_{31}) \\ a_3 &= \tfrac{1}{2s}(Q_{21}-Q_{12}) \end{aligned}$$

Spectral Decomposition Theorem

Let $\mathbf{S}$ be a symmetric tensor. Then there is an orthonormal basis for $\mathcal{V}$ consisting entirely of eigenvectors of $\mathbf{S}$. Moreover, for any such basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the corresponding eigenvalues ω_1, ω_2, and ω_3, when ordered, form the entire spectrum of $\mathbf{S}$ and

$$\mathbf{S} = \sum_{i=1}^{3} \omega_i\, \mathbf{e}_i \otimes \mathbf{e}_i.$$

Conversely, if $\mathbf{S}$ has the form

$$\mathbf{S} = \sum_{i=1}^{3} \omega_i\, \mathbf{e}_i \otimes \mathbf{e}_i,$$

with $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ orthonormal, then ω_1, ω_2, and ω_3 are eigenvalues of $\mathbf{S}$ with $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ being the corresponding eigenvectors. Furthermore,

(a) $\mathbf{S}$ has exactly three distinct eigenvalues if and only if the characteristic spaces of $\mathbf{S}$ are three mutually perpendicular lines through $\mathbf{0}$.

(b) $\mathbf{S}$ has exactly two distinct eigenvalues if and only if $\mathbf{S}$ admits the representation

$$\mathbf{S} = \omega_1\, \mathbf{e} \otimes \mathbf{e} + \omega_2\, (\mathbf{I} - \mathbf{e} \otimes \mathbf{e}), \quad \|\mathbf{e}\| = 1 \text{ and } \omega_1 \neq \omega_2.$$

In this case ω_1 and ω_2 are two distinct eigenvalues and the corresponding characteristic spaces are span$\{\mathbf{e}\}$ and $\{\mathbf{e}\}^\perp$, respectively. Conversely, if span$\{\mathbf{e}\}$ and $\{\mathbf{e}\}^\perp$ (with $\|\mathbf{e}\| = 1$) are the characteristic spaces of $\mathbf{S}$ then $\mathbf{S}$ must have the form

$$\mathbf{S} = \omega_1\, \mathbf{e} \otimes \mathbf{e} + \omega_2\, (\mathbf{I} - \mathbf{e} \otimes \mathbf{e}), \quad \omega_1 \neq \omega_2.$$

(c) $\mathbf{S}$ has exactly one eigenvalue if and only if

$$\mathbf{S} = \omega\, \mathbf{I}.$$

In this case ω is the eigenvalue and $\mathcal{V}$ is the corresponding characteristic space. Conversely, if $\mathcal{V}$ is a characteristic space for $\mathbf{S}$ then $\mathbf{S}$ has the form

$$\mathbf{S} = \omega\, \mathbf{I}.$$

Polar Decomposition Theorem

Let $\mathbf{F} \in \text{Lin}^+$, i.e., let $\mathbf{F}$ be a second order tensor with positive determinant. Then there exists positive definite symmetric tensor $\mathbf{U}$ and $\mathbf{V}$ along with a proper rotation $\mathbf{R}$ such that

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}.$$

Moreover, each of these decompositions is unique; in fact

$$\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}} \quad \text{and} \quad \mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T}.$$

We call the representations $\mathbf{F} = \mathbf{UR}$ and $\mathbf{F} = \mathbf{RV}$ the right and left polar decompositions of $\mathbf{F}$, respectively.

Principal Invariants of a Tensor

Given a tensor $\mathbf{S}$, the determinant of the tensor $\mathbf{S} - \omega\mathbf{I}$ admits the representation

$$\det(\mathbf{S} - \omega\, \mathbf{I}) = -\omega^3 + \mathcal{I}_1(\mathbf{S})\, \omega^2 - \mathcal{I}_2(\mathbf{S})\, \omega + \mathcal{I}_3(\mathbf{S}) \quad \forall \omega \in \mathbb{R},$$

where

$$\begin{aligned} \mathcal{I}_1(\mathbf{S}) &= \text{tr}\,(\mathbf{S})\,, \\ \mathcal{I}_2(\mathbf{S}) &= \tfrac{1}{2}\left((\text{tr}(\mathbf{S}))^2 - \text{tr}\left(\mathbf{S}^2\right)\right), \\ \mathcal{I}_3(\mathbf{S}) &= \det(\mathbf{S})\,. \end{aligned}$$

The elements of the list $\{\mathcal{I}_1(\mathbf{S}), \mathcal{I}_2(\mathbf{S}), \mathcal{I}_3(\mathbf{S})\}$ are called the *principal invariants* of $\mathbf{S}$. Finally, if $\mathbf{S}$ is symmetric, its principal invariants are completely characterized by the spectrum of $\mathbf{S}$. In fact, we have

$$\begin{aligned} \mathcal{I}_1(\mathbf{S}) &= \omega_1 + \omega_2 + \omega_3, \\ \mathcal{I}_2(\mathbf{S}) &= \omega_1\omega_2 + \omega_2\omega_3 + \omega_1\omega_3, \\ \mathcal{I}_3(\mathbf{S}) &= \omega_1\omega_2\omega_3, \end{aligned}$$

ω_1, ω_2, and ω_3 being the eigenvalues of $\mathbf{S}$.

Cayley-Hamilton Theorem

Recall that the equation

$$\det(\mathbf{S} - \omega \mathbf{I}) = 0$$

is called the characteristic equation of $\mathbf{S}$.

Every second order tensor $\mathbf{S}$ satisfies its own characteristic equation, that is,

$$\mathbf{S}^3 - \mathcal{I}_1(\mathbf{S})\,\mathbf{S}^2 + \mathcal{I}_2(\mathbf{S})\,\mathbf{S} - \mathcal{I}_3(\mathbf{S})\mathbf{I} = \mathbf{0}.$$

Tensor Analysis

Useful Differentiation Formulas

Let $\mathbf{F}$, $\mathbf{U}$, and $\mathbf{S}$ be smooth tensor fields over. Let $\mathbf{u}$ and $\mathbf{v}$ be smooth vector fields. Finally, let ϕ be a smooth scalar field. Then,

$$
\begin{aligned}
D \det(\mathbf{F})[\mathbf{U}] &= \det(\mathbf{F}) \operatorname{tr}\left(\mathbf{U}\mathbf{F}^{-1}\right), \\
\operatorname{div} \mathbf{v} &= \operatorname{tr}(\operatorname{grad} \mathbf{v}), \\
\operatorname{grad}(\phi \mathbf{v}) &= \phi \operatorname{grad} \mathbf{v} + \mathbf{v} \otimes \operatorname{grad} \phi, \\
\operatorname{div}(\phi \mathbf{v}) &= \phi \operatorname{div} \mathbf{v} + \mathbf{v} \cdot \operatorname{grad} \phi, \\
\operatorname{grad}(\mathbf{v} \cdot \mathbf{u}) &= (\operatorname{grad} \mathbf{u})^T \mathbf{v} + (\operatorname{grad} \mathbf{v})^T \mathbf{u}, \\
\operatorname{div}(\mathbf{v} \otimes \mathbf{u}) &= \mathbf{v} \operatorname{div} \mathbf{u} + (\operatorname{grad} \mathbf{v}) \mathbf{u} \\
\operatorname{div}\left(\mathbf{S}^T \mathbf{v}\right) &= \mathbf{S} \cdot \operatorname{grad} \mathbf{v} + \mathbf{v} \cdot \operatorname{div} \mathbf{S} \\
\operatorname{div}(\phi \mathbf{S}) &= \phi \operatorname{div} \mathbf{S} + \mathbf{S} \operatorname{grad} \phi
\end{aligned}
$$

Divergence Theorem

Let B be a regular region of a Euclidean space $\mathcal{E}$. Let ∂B be the boundary of B with outer unit normal $\mathbf{n}$. Let $\mathbf{S}$ be a smooth tensor fields over B. Let $\mathbf{v}$ and $\mathbf{w}$ be smooth vector fields over B. Finally, let ϕ be a smooth scalar field over B. Then,

$$
\begin{aligned}
\int_{\partial B} \phi \, \mathbf{n} \, ds &= \int_B \operatorname{grad} \phi \, dv \\
\int_{\partial B} \mathbf{v} \cdot \mathbf{n} \, ds &= \int_B \operatorname{div} \mathbf{v} \, dv \\
\int_{\partial B} \mathbf{S}\mathbf{n} \, ds &= \int_B \operatorname{div} \mathbf{S} \, dv \\
\int_{\partial B} \mathbf{v} \otimes \mathbf{n} \, ds &= \int_B \operatorname{grad} \mathbf{v} \, dv \\
\int_{\partial B} (\mathbf{S}\mathbf{n}) \otimes \mathbf{v} \, ds &= \int_B \left[(\operatorname{div} \mathbf{S}) \otimes \mathbf{v} + \mathbf{S} (\operatorname{grad} \mathbf{v})^T\right] dv \\
\int_{\partial B} \mathbf{v} \cdot \mathbf{S}\mathbf{n} \, ds &= \int_B (\mathbf{v} \cdot \operatorname{div} \mathbf{S} + \mathbf{S} \cdot \operatorname{grad} \mathbf{v}) \, dv \\
\int_{\partial B} \mathbf{v}(\mathbf{w} \cdot \mathbf{n}) \, ds &= \int_B [\mathbf{v} \operatorname{div} \mathbf{w} + (\operatorname{grad} \mathbf{v})\mathbf{w}] \, dv
\end{aligned}
$$