1. (**15 Points**) Express the matrices $\mathbf{C} = \mathbf{AB}, \mathbf{D} = \mathbf{BA}, \mathbf{E} = \mathbf{AB}^{\mathrm{T}}$ in terms of matrix components A_{ij}, B_{ij}.
   Hint: You may need to change the indices of **A** and **B**.

2. (**4 × 15 = 60 Points**). Symmetry and antisymmetry,
   (a) Let *symmetric* and *antisymmetric* parts of a matrix be defined as,

$$A_{(ij)} := \frac{1}{2}\left(A_{ij} + A_{ji}\right) \tag{1a}$$

$$A_{[ij]} := \frac{1}{2}\left(A_{ij} - A_{ji}\right) \tag{1b}$$

$$\tag{1c}$$

   Show that $A_{ij}B_{ji} = A_{(ij)}B_{(ji)} + A_{[ij]}B_{[ji]}$

   (b) If **S** is symmetric ($\mathbf{S}^{\mathrm{T}} = \mathbf{S}$) and **W** is antisymmetric ($\mathbf{W}^{\mathrm{T}} = -\mathbf{W}$), show that $S_{ij}W_{ij} = S_{ij}W_{ji} = 0$.

   (c) If a matrix **A** is symmetric and antisymmetric, show it is zero: $\mathbf{A} = 0$.

   (d) Show that any matrix **A** can be *uniquely* written as the sum of a symmetric and an antisymmetric matrices.
   Hint: Consider $\mathbf{S} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathrm{T}})$, $\mathbf{W} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{T}})$. For uniqueness assume $\mathbf{A} = \tilde{\mathbf{S}} + \tilde{\mathbf{W}}$ where $\tilde{\mathbf{S}}$ is symmetric and $\tilde{\mathbf{W}}$ is antisymmetric. Show $\tilde{\mathbf{S}} = \mathbf{S}$ and $\tilde{\mathbf{W}} = \tilde{\mathbf{W}}$. Use previous problem for the latter proof.

3. (**20 Points**) Starting from,

$$\det \mathbf{A} = \epsilon_{ijk}A_{1i}A_{2j}A_{3k} \tag{2}$$

   which was proven in the class, prove the following equation:

$$\epsilon_{pqr}\det \mathbf{A} = \epsilon_{ijk}A_{pi}A_{qj}A_{rk} \tag{3}$$

   Hint: Consider $p = 1, q = 2, r = 3$ and from that consider all 5 other cases that p, q, r are distinct. From (2) these correspond to determinant of matrices whose rows are permutations of matrix **A**. Use the fact that odd and even number of permutations of rows multiply the determinant of a matrix by -1 and 1, respectively. Finally, for cases that at least two of p, q, r are equal, use a fact about the determinant of a matrix whose two or more rows are equal.

4. (**30 Points**) By only using indicial notation show (do not expand in explicit components in terms of indices $1, 2, 3$),

$$\det(\mathbf{AB}) = (\det \mathbf{A}).(\det \mathbf{B}) \tag{4}$$

   Next, using this equation show that for inverse of **A**, $\mathbf{A}^{-1}$, satisfying $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ (**I** is the identity matrix),

$$\det \mathbf{A}^{-1} = 1/\det \mathbf{A} \tag{5}$$

   Hint: For $\mathbf{C} = \mathbf{AB}$ use,

$$\det \mathbf{C} = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}C_{ip}C_{jq}C_{kr}$$

   expand components of **C** in terms of components of **A** and **B**, and finally use one of the two equations from previous question.

5. (**20 Points**) Show that the components of the inverse of a matrix satisfy,

$$A^{-1}_{rk} = \frac{1}{2 \det A} \epsilon_{ijk} \epsilon_{pqr} A_{ip} A_{jq} \tag{6}$$

Hint: Compute $A^{-1}_{rk} A_{km}$.

6. (**25 (a,b) + 30 - extra-credit (c,d) = 55 Points**). Examples of function vector spaces.

For the functions defined on the interval $[a\ b]$ define the operator $\langle f, g \rangle = \int_a^b = f(x)g(x)\,\mathrm{d}x$. Provided the integral exists and is finite. We define the L2 space of integrable functions on $[a\ b]$ as $L^2([a\ b]) = \{| \langle f, f \rangle | < \infty\}$. That is, $L^2([a\ b])$ is the space of square integrable functions on $[a\ b]$. For brevity we define $\mathcal{V} := L^2([a\ b])$

(a) Show that $\mathcal{V}$ is a vector space with the usual function addition and scalar function multiplication definitions.

(b) Show $\langle ., . \rangle$ is an inner product on $\mathcal{V}$ so $\mathcal{V}$ is an inner-product vector space.

(c) Is the operator $||.||$,

$$||.|| : \mathcal{V} \to \mathbb{R}^+ \cup \{0\} \ni \forall f \in \mathcal{V} \quad ||f|| := \sqrt{\langle f, f \rangle}$$

a norm on $\mathcal{V}$? Is $(\mathcal{V}, ||.||)$ a *normed vector space*?

(d) Let us define

$$||.||_p : \mathcal{V}_p \to \mathbb{R}^+ \cup \{0\} \ni \forall f \in \mathcal{V}_p \quad ||f||_p := \sqrt[p]{\int_a^b |f(x)|^p \,\mathrm{d}x}$$

where $\mathcal{V}_p = \{f | \int_a^b |f(x)|^p < \infty\}$ is the space of power p integrable functions which is denoted by $L_p([a\ b])$. One can show that $(\mathcal{V}_p, ||.||_p)$ is a *normed vector space*. We skip the proof herein. In the limit $p \to \infty$ the norm converges to $||.||_\infty$ norm,

$$||.||_\infty : \mathcal{V}_\infty \to \mathbb{R}^+ \cup \{0\} \ni \forall f \in \mathcal{V}_\infty \quad ||f||_\infty = \mathrm{Max}_{x \in \mathcal{V}_\infty}(|f(x)|)$$

where $\mathcal{V}_\infty$ is the space of bounded functions on $[a\ b]$. Again, $(\mathcal{V}_\infty, ||.||_\infty)$ is a normed vector space. Are any of these $L_p([a\ b])$ spaces ($p > 2$, including ∞) an inner-product space in addition of a normed space? Can we claim that a normed vector space is also an inner-product vector space? How about the reverse?