1. **Axis and eigenvalues of an orthogonal transformation.**(**30 Points**)
   (a) If λ is an eigenvalue of an orthogonal tensor $\mathbf{Q}$ show that $|\lambda| = 1$.
   (b) To complete the determination of eigenvalues of orthogonal tensors show that
      i. A proper orthogonal tensor has an eigenvalue of 1. The corresponding eigenvector $\mathbf{u}$ is called the axis of $\mathbf{Q}$.
      Hint: For an eigenvalue of λ we have $\det(\mathbf{Q} - \lambda\mathbf{I}) = 0$. So, for $\lambda = 1$ we need to show $\det(\mathbf{Q} - \mathbf{I}) = 0$. To do this, expand $\det[\mathbf{Q}^{\mathrm{T}}(\mathbf{Q} - \mathbf{I})]$ two ways: One use determinant of product rule and the other evaluate the determinant of the term directly, use $\det \mathbf{Q} = 1$ (proper orthogonal) to show $\det(\mathbf{Q} - 1\mathbf{I}) = 0$.
      ii. Show that an improper orthogonal tensor has an eigenvalue of -1. Again, the corresponding eigenvector $\mathbf{u}$ is called the axis of $\mathbf{Q}$.
      Hint: Use a similar proof to proper orthogonal tensors, but this time use $\mathbf{Q} + \mathbf{I}$.
   (c) We will eventually show that an orthogonal tensor can be written as,

$$\mathbf{Q} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \tag{1}$$

   where $+$ and $-$ signs correspond to proper and improper orthogonal tensors, respectively, and direction $\mathbf{e}_3$ is aligned with the axis of $\mathbf{Q}$, $\mathbf{Q}$.
   We already discussed that one eigenvalue is ± 1 which corresponds to eigenvector $\mathbf{u} = \mathbf{e}_3$. By using $\det(\mathbf{Q} - \lambda\mathbf{I}) = 0$, as the equation of the eigenvalue of $\mathbf{Q}$ show that in fact ± 1 is the only real eigenvalue for a proper/improper orthogonal tensor.

2. **Orthonormality condition** (**10 Points**): Using $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$ show that vectors formed by columns (which are $\mathbf{c}^i = \mathbf{Q}\mathbf{e}_i$ and rows (which are formed by $\mathbf{r}^i = \mathbf{Q}^{\mathrm{T}}\mathbf{e}_i$) are orthonormal: $\mathbf{r}^i.\mathbf{r}^j = \delta^{ij}$, $\mathbf{c}^i.\mathbf{c}^j = \delta^{ij}$.

3. **Representation of an orthogonal tensor** (**70 Points**): To obtain (1) we start with a representation of $\mathbf{Q}$ such that axis of $\mathbf{Q}$ is aligned with coordinate system $\mathbf{e}_3$ unit vector as shown in fig. 3. The components of $\mathbf{Q}$ in this orthonormal coordinate system is shown below:

$$\mathbf{Q} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \tag{2}$$

   (a) Show that the last column of $\mathbf{Q}$ is $[0\ 0\ \pm 1]^{\mathrm{T}}$ (+ for proper orthogonal and - for improper orthogonal).
   Hint: Since $\mathbf{e}_3$ is aligned with the axis of $\mathbf{Q}$ it is an eigenvector with eigenvalue of ± 1. Use the fact that column 3 ($\mathbf{c}^3$) is the image of $\mathbf{Q}$ on $\mathbf{e}_3$ ($\mathbf{Q}\mathbf{e}_3$).
   (b) Show that $Q_{31} = Q_{32} = 0$ (unknowns of the third row). After these two steps $\mathbf{Q}$ looks like:

$$\mathbf{Q} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \tag{3}$$

   Hint: Use orthonormality of columns (or rows ?) of $\mathbf{Q}$.
   (c) Show that $|Q_{11}| \leq 1$, again using orthonormality property, and subsequently show that an angle θ exists such that $Q_{11} = \cos(\theta), Q_{21} = \sin(\theta)$.

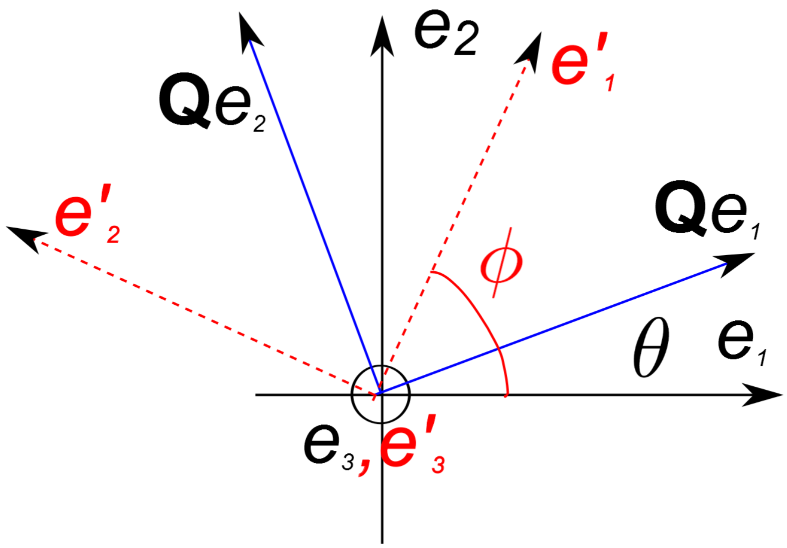

Figure 1: A coordinate system such that $\mathbf{e}_3$ is aligned with the axis of an orthogonal tensor $\mathbf{Q}$.

(d) Finally, show that $Q_{12} = -\sin(\theta)$ and $Q_{22} = \cos(\theta)$.
Hint: Again use orthonormality condition. This will not determine Q_{i2} with a factor ± 1. What condition should we use to further narrow down these two values to the one given?

(e) After plugging remaining values of Q_{ij}, we obtain (1). Discuss why the values $\mathbf{Qe}_1$, $\mathbf{Qe}_2$ shown in the figure, which correspond to rotation with angle θ, are consistent with $\mathbf{Qe}_1$, $\mathbf{Qe}_2$ from (1).
Hint: Use the fact that columns of a tensors are images of the tensor on unit vectors of the corresponding coordinate system.

(f) What is the image of this orthonormal tensor on $\mathbf{e}_3$ for proper and improper cases?

(g) From previous two questions, summarize what proper / improper orthogonal tensors represent.

4. **Axis and angle of rotation** (**60 Points**): From previous question we observe an orthogonal tensor is a rotation plus possibly another operation (to be specified by you). The question is how to determine the axis (of rotation) and angle of rotation. If the tensor is represented in a coordinate system such that $\mathbf{e}_3$ is aligned with axis of $\mathbf{Q}$, denoted by ax($\mathbf{Q}$),the answer is easy, but we want to answer these questions in general case. You can assume $\mathbf{Q}$ is only a rotation (*i.e.,* proper orthogonal $\det \mathbf{Q} = 1$). I will present two approaches: Method A will follow from the development above but has a pitfall, and Method B addresses that issue.

   (a) Method A (Using eigenvalues and one of the "fundamental invariants" of $\mathbf{Q}$):

      i. Using the eigenvalue analysis above show how we can obtain the axis of rotation ax($\mathbf{Q}$).

      ii. By looking at "fundamental invariants" of tensor $\mathbf{Q}$ find one that can determine the angle of rotation.
      Hint: You may refer to the representation of $\mathbf{Q}$ in the coordinate system where $\mathbf{e}_3$ is aligned with ax($\mathbf{Q}$), *cf.* (1), and evaluate fundamental invariants for that.

      iii. Can we uniquely identify angle of rotation θ or we obtain it with a sign ambiguity in the form $\pm\theta$ using the fundamental invariant chosen? What is the interpretation of this? Do we need to change the axis of rotation if the sign of the angle is reversed?

(b) Method B (Using skew part of $\mathbf{Q}$). This approach is more robust and provides the angle of rotation with no ambiguity.

i. Show that given the angle of rotation θ and axis of rotation $\text{ax}(\mathbf{Q}) := \mathbf{a}$, we can express $\mathbf{Q}$ as.

$$\mathbf{Q} = \cos\theta \mathbf{I} + (1 - \cos\theta)\mathbf{a} \otimes \mathbf{a} + \sin\theta \text{ax}(\mathbf{a}) \tag{4}$$

where ax$\mathbf{a}$ is the skew-symmetric tensor formed by $\mathbf{a}$. **Hint**: Try to demonstrate this for **one** coordinate system (*e.g.,* when $\mathbf{e}_3$ is aligned with $\text{ax}(\mathbf{Q})$) then given that both sides are tensor (follow tensorial transformation rules) we can claim showing (4) irrespective to the choice of coordinate system.
Equation (4) is a very useful identity for expressing $\mathbf{Q}$ in a given coordinate when angle, and axis of rotation are provided in that coordinate system.

ii. Taking the trace of both sides of the equation (4) obtain an equation for the angle θ under the condition $0 \le \theta < \pi$. This will uniquely identify the value $\sin(\theta) \ge 0$.

iii. By taking the transpose of (4) and forming skew($\mathbf{Q}$) show that,

$$\mathbf{a} = \text{ax}(\mathbf{Q}) = \frac{1}{\sin\theta}\text{ax}\left[\text{skew}(\mathbf{Q})\right] \tag{5}$$

thus a way to uniquely identify θ and $\text{ax}(\mathbf{Q})$ without the sign ambiguity of the first approach.

Note: In the matlab code provided in the next question, you will observe the application of both methods discussed in this question.

5. **Product of orthogonal tensors(60 Points)**: If $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are orthonormal tensors $\mathbf{Q}_2\mathbf{Q}_1$ means we first apply $\mathbf{Q}_1$ on a vector followed by $\mathbf{Q}_2$ acting on the resultant from the first operation.

(a) Fill out the following sentences (only provide words for V1 to V5 in your answers).
Product of two orthogonal tensors is a(an) $\cdots$V1$\cdots$ tensor representing one $\cdots$V2$\cdots$ operation and a $\cdots$V3$\cdots$ operation when the resultant is an $\cdots$V4$\cdots$ tensor.
Product of two proper orthogonal tensors is a(an) $\cdots$V5$\cdots$ tensor, representing only one $\cdots$V6$\cdots$ operation.
Hint: Given the orthogonality of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ investigate if $\mathbf{Q}_1\mathbf{Q}_2$ has any special tensor property.

(b) Consider the following two rotations,

$$\mathbf{Q}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & -\sin(\theta_1) \\ 0 & \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}, \qquad \mathbf{Q}_2 = \begin{bmatrix} \cos(\theta_2) & 0 & -\sin(\theta_2) \\ 0 & 1 & 0 \\ \sin(\theta_2) & 0 & \cos(\theta_2) \end{bmatrix} \tag{6}$$

for $\theta_1 = \frac{\pi}{6}$ and $\theta_2 = \frac{\pi}{3}$. By running the matlab code http://rezaabedi.com/teaching/continuum-mechanics/HW3.m answer the following questions:

i. What are the axis and angle of rotation for $\mathbf{Q}_1\mathbf{Q}_2$?

ii. What are the axis and angle of rotation for $\mathbf{Q}_2\mathbf{Q}_1$?

iii. Are the answers the same for $\mathbf{Q}_1\mathbf{Q}_2$ and $\mathbf{Q}_1\mathbf{Q}_2$? [1].

iv. Once you identify axis and angle of rotation for $\mathbf{Q}_2\mathbf{Q}_1$, reconstruct it using (4) and verify that the reconstruction based on its angle and axis matches the original $\mathbf{Q}_2\mathbf{Q}_1$.

v. Can you think of rotations that are commutative?
Hint: Think about (1) for two θ_1 and θ_2.

[1] FYI: Using the traces of $\mathbf{Q}_1\mathbf{Q}_2$ and $\mathbf{Q}_2\mathbf{Q}_1$ what can we say about the angles of rotation for these two tensors? No need to submit this answer

6. **Coordinate transformation of a rotation tensor(20 Points)**: One should be careful in distinguish between an orthogonal tensor and coordinate transformation. For example, let us consider $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$, specified by rotation of angle ϕ with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, along $\mathbf{e}_3$ axis as shown in fig. 3. The coordinate transformation **MATRIX** is given by,

$$\mathbf{Q}_{\text{t}r} = \begin{bmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

Express $\mathbf{Q}$ with respect to $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ coordinate system, whose components with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ coordinate system are given in (1).

7. **Exact and small-angle rotation tensors(50 Points)**: Consider the equation (1), where the rotation angle θ is time-dependent and is specified by a constant angular speed ω: $\theta = \omega t$. Thus, $\mathbf{Q}$ is represented as a function of time $\mathbf{Q}(t)$. A vector with position $\mathbf{u}$ at initial time is rotated with $\mathbf{Q}$ and takes position $\mathbf{u}(t) = \mathbf{Q}(t)u(0)$ for all times. We want to compare exact rotation and and small angle rotation changes between t and $t + \mathit{\Delta} t$ for the vector $\mathbf{u}$:

$$\mathit{\Delta} u(t, \mathit{\Delta} t) = \mathbf{u}(t + \mathit{\Delta} t) - \mathbf{u}(t) = \{\mathbf{Q}(t + \mathit{\Delta} t) - \mathbf{Q}(t)\}\, \mathbf{u}(0) \qquad \text{Exact rotation change} \tag{8a}$$

$$\delta u(t, \mathit{\Delta} t) = \left(\mathit{\Delta} t \dot{\mathbf{Q}}(t)\right) \mathbf{u}(0) \qquad \text{Small-angle rotation} \tag{8b}$$

Equation (8b) is obtained from (8a) by Taylor expansion of $\mathbf{Q}(t + \mathit{\Delta} t)$ around t and ignoring second order and higher terms (basically using derivative approximation). For both equations consider $t = 0$ and $\mathit{\Delta} t$ is a small angle (as needed for (8b)). Thus, we simply use $\mathit{\Delta} u(\mathit{\Delta} t)$ and $\delta u(\mathit{\Delta} t)$ and rewrite rotation change relative to initial position,

$$\mathit{\Delta} u(\mathit{\Delta} t) = \mathbf{u}(\mathit{\Delta} t) - \mathbf{u}(0) = \{\mathbf{Q}(t) - \mathbf{Q}(0)\}\, \mathbf{u}(0) \qquad \text{Exact rotation change} \tag{9a}$$

$$\delta u(\mathit{\Delta} t) = \left(\mathit{\Delta} t \dot{\mathbf{Q}}(0)\right) \mathbf{u}(0) \qquad \text{Small-angle rotation} \tag{9b}$$

(a) Show that $\dot{\mathbf{Q}}(0)$ is skew-symmetric.
Hint: Take derivative of $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$, then plug $t = 0$ and note that $\mathbf{Q}(0) = \mathbf{I}$.

(b) Express the tensorial notation of small angle rotation $\mathbf{W} := \left(\mathit{\Delta} t \dot{\mathbf{Q}}(0)\right)$ for $\theta = \omega t$ from (9b).

(c) Obtain the same small angle rotation tensor by Taylor expansion of $\sin(\theta)$ and $\cos(\theta)$ in $\{\mathbf{Q}(t) - \mathbf{Q}(0)\}$ from (9a) and verify that it matches the previous result.

(d) Express axial vector $\mathbf{w} = \mathrm{ax}(\mathbf{W})$. What is the direction of $\mathbf{w}$? How is this direction related to the axis of $\mathbf{Q}$? What is its magnitude?
Hint: Compare $\mathrm{ax}(\mathbf{W})$ and $\mathrm{ax}(\mathbf{Q})$ with equation (5).

(e) In a schematic, draw the change of exact rotation $\mathbf{Q}$ and small-angle rotation $\mathbf{W}$ for a vector that has components both parallel to the axis of rotation and also perpendicular to it. Briefly (less than 3 sentences) discuss how these values are different.