1. (**$4 \times 20 = 80$ Points**) **Lagrangian and Eulerian strains**: Remember that right and left polar decompositions of $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$,

$$\mathbf{U} = \sqrt{C} = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i \qquad \text{Right stretch (useful Referential / Lagrangian conf.)} \tag{1a}$$

$$\mathbf{V} = \sqrt{B} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i \qquad \text{Left stretch (useful Spatial / Eulerian conf.)} \tag{1b}$$

where $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ and $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, λ_i are eigenvalues of both $\mathbf{U}, \mathbf{V}$ (why the have the same eigenvalues?) and $\mathbf{u}_i, \mathbf{v}_i = \mathbf{R}\mathbf{u}_i$ are orthonormal eigenvectors of $\mathbf{U}, \mathbf{V}$. General Referential (Lagrangian) $\boldsymbol{\varepsilon}$ and Spatial (Eulerian) $\boldsymbol{\varepsilon}^*$ strains are defined by,

$$\epsilon_e := e(\mathbf{U}) = \sum_{i=1}^{3} e(\lambda_i) \mathbf{u}_i \otimes \mathbf{u}_i \qquad \text{Referential (Lagrangian)} \tag{2a}$$

$$\epsilon_e^* := e(\mathbf{V}) = \sum_{i=1}^{3} e(\lambda_i) \mathbf{v}_i \otimes \mathbf{v}_i \qquad \text{Spatial (Eulerian)} \tag{2b}$$

for the function e satisfying,

$$e(1) = 0 \tag{3a}$$

$$e'(1) = 1 \tag{3b}$$

$$e'(\lambda) > 0 \quad \text{for all } \lambda > 0 \quad \text{(so } e \text{ is an increasing function} \tag{3c}$$

Lagrangian strain approximates (or is equal to if $e(\lambda) = \lambda - 1$) $(|\mathrm{d}\mathbf{y}| - |\mathrm{d}\mathbf{x}|)/|\mathrm{d}\mathbf{x}|$ while Eulerian strain approximates (or is equal to if $e(\lambda) = 1 - \lambda^{-1}$) $(|\mathrm{d}\mathbf{y}| - |\mathrm{d}\mathbf{x}|)/|\mathrm{d}\mathbf{y}|$. To circumvent calculating square roots $\mathbf{U}$ and $\mathbf{V}$ we can use $e_g(\lambda) = (\lambda^2 - 1)/2 \approx (\lambda - 1)$ for $\lambda \approx 1$ for Lagrangian and $e_{g^*}(\lambda) = (1 - \lambda^{-2})/2 \approx 1 - \lambda^{-1}$ (for $\lambda \approx 1$).

(a) Show that Strain tensors corresponding to e_g and e_{g^*} are Green $\mathbf{G}$ and Almansi $\mathbf{G}^*$ tensors,

$$\mathbf{G} := e_g(\mathbf{U}) = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \tag{4a}$$

$$\mathbf{G}^* := e_{g^*}(\mathbf{V}) = \frac{1}{2}\left(\mathbf{I} - \mathbf{B}^{-1}\right) \tag{4b}$$

(b) Consider reference differential (change of) position $\mathrm{d}\mathbf{x}$ and its spatial (change of) position $\mathrm{d}\mathbf{y}$ related by $\mathrm{d}\mathbf{y} = \mathbf{F}\mathrm{d}\mathbf{x}$. By direct substitution $\mathrm{d}\mathbf{y} = \mathbf{F}\mathrm{d}\mathbf{x}$ (*i.e.*, you cannot simply use (4)) show that,

$$\varepsilon_g(\mathbf{x}, \mathbf{e}_x) = \frac{1}{2}\frac{|\mathrm{d}\mathbf{y}|^2 - |\mathrm{d}\mathbf{x}|^2}{|\mathrm{d}\mathbf{x}|^2} = \left(\frac{|\mathrm{d}\mathbf{y}| - |\mathrm{d}\mathbf{x}|}{|\mathrm{d}\mathbf{x}|}\right)\left(\frac{|\mathrm{d}\mathbf{y}| + |\mathrm{d}\mathbf{x}|}{2|\mathrm{d}\mathbf{x}|}\right) = \mathbf{e}_x.\mathbf{G}\mathbf{e}_x, \quad \mathbf{e}_x = \frac{\mathrm{d}\mathbf{x}}{|\mathrm{d}\mathbf{x}|} \tag{5a}$$

$$\varepsilon_{g^*}(\mathbf{y}, \mathbf{e}_y) = \frac{1}{2}\frac{|\mathrm{d}\mathbf{y}|^2 - |\mathrm{d}\mathbf{x}|^2}{|\mathrm{d}\mathbf{y}|^2} = \left(\frac{|\mathrm{d}\mathbf{y}| - |\mathrm{d}\mathbf{x}|}{|\mathrm{d}\mathbf{y}|}\right)\left(\frac{|\mathrm{d}\mathbf{y}| + |\mathrm{d}\mathbf{x}|}{2|\mathrm{d}\mathbf{y}|}\right) = \mathbf{e}_y.\mathbf{G}^*\mathbf{e}_y, \quad \mathbf{e}_y = \frac{\mathrm{d}\mathbf{y}}{|\mathrm{d}\mathbf{y}|} \tag{5b}$$

where $\varepsilon_g(\mathbf{x}, \mathbf{e}_x)$ is the referential (Lagrangian) strain for direction $\mathbf{e}_x$ and position $\mathbf{x}$ in referential configuration and function e_g; similarly $\varepsilon_{g^*}(\mathbf{y}, \mathbf{e}_y)$ is the spatial (Eulerian) strain for direction $\mathbf{e}_y$ and position $\mathbf{y}$ in spatial configuration and function e_{g^*}.

(c) In referential and spatial configurations functions are represented in terms of $\mathbf{x}$ and $\mathbf{y}$, respectively. Strains involve (referential or spatial) gradients of deformation/displacement. Referential ($\mathbf{H}$) and spatial ($\mathbf{H}^*$) *displacement gradients* are

$$\mathbf{H} = \nabla_{\mathbf{x}}\mathbf{u}, \qquad \text{that is} \quad H_{ij} = \frac{\partial u_i(\mathbf{x}, t)}{\partial x_j} \tag{6a}$$

$$\mathbf{H}^* = \nabla_{\mathbf{y}}\mathbf{u}, \qquad \text{that is} \quad H^*_{ij} = \frac{\partial u_i(\mathbf{y}, t)}{\partial y_j} \tag{6b}$$

where $\mathbf{u} = \mathbf{y} - \mathbf{x}$ is displacement field. Show the following in order,

$$\mathbf{H} = \mathbf{F} - \mathbf{I} \tag{7a}$$

$$\mathbf{H} = \mathbf{H}^*\mathbf{F} \tag{7b}$$

$$\mathbf{H}^* = \mathbf{I} - \mathbf{F}^{-1} \quad \text{directly by } \mathbf{u} = \mathbf{y} - \mathbf{x} \text{ (preferred) or using the last two equations} \tag{7c}$$

(d) Expansion of $\mathbf{G}$ and $\mathbf{G}^*$: Using (4) and (9) show that,

$$\mathbf{G} := \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}}\mathbf{H}\right) \Rightarrow \tag{8a}$$

$$G_{ij} := \frac{1}{2}\left(H_{ij} + H_{ji} + H_{ki}H_{kj}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

$$\mathbf{G}^* := \frac{1}{2}\left(\mathbf{H}^* + \mathbf{H}^{*\mathrm{T}} - \mathbf{H}^{*\mathrm{T}}\mathbf{H}^*\right) \Rightarrow \tag{8b}$$

$$G^*_{ij} := \frac{1}{2}\left(H^*_{ij} + H^*_{ji} - H^*_{ki}H^*_{kj}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i} - \frac{\partial u_k}{\partial y_i}\frac{\partial u_k}{\partial y_j}\right)$$

Note: As we will observe *Cauchy stress tensor* $\mathbf{T}(\mathbf{y}, t)$ is naturally expressed in spatial coordinate and the equation of motion $\mathrm{div}T + \rho_0 b = \frac{\mathrm{D}\rho_0 v}{\mathrm{D}t}$ is also written for spatial configuration. The operator "div" is spatial divergence $\mathrm{div}T = \frac{\partial T_{ij}(\mathbf{y},t)}{\partial y_j}$. Clearly, in spatial configuration, $\mathbf{T}$ expressed in terms of a spatial strain measure such as $\mathbf{G}^*$ in (8b) simplifies the equation of motion as eventually two level derivatives on $\mathbf{u}$ will be all on spatial coordinate $\mathbf{y}$. On the other hand, by pulling $\mathbf{T}$ back to referential configuration, *i.e., Piola-Kirchhoff* stress tensors, we will be dealing with referential divergence (Div) and strain measures with referential displacement gradient, such as $\mathbf{G}$ (8b), are the appropriate choice. In solid mechanics, we often express equations in referential configuration.

2. (**5 × 20 = 100 Points**) **Small deformation gradient**: Let us consider a infinitesimal deformation gradient problem, *i.e.,* $\mathbf{H} = \mathcal{O}(\epsilon), \epsilon \ll 1$. We define *linearized* strains,

$$\mathbf{E} := \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}}\right) \Rightarrow E_{ij} = \frac{1}{2}\left(H_{ij} + H_{ji}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{9a}$$

$$\mathbf{E}^* := \frac{1}{2}\left(\mathbf{H}^* + \mathbf{H}^{*\mathrm{T}}\right) \Rightarrow E^*_{ij} = \frac{1}{2}\left(H^*_{ij} + H^*_{ji}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i}\right) \tag{9b}$$

Clearly, $\mathbf{G} = \mathbf{E} + \mathcal{O}(\epsilon^2)$ and $\mathbf{G}^* = \mathbf{E}^* + \mathcal{O}(\epsilon^2)$. We want to show that in fact <u>all strains defined by</u> (2) <u>are equal to $\mathbf{E}, \mathbf{E}^*$ within $\mathcal{O}(\epsilon^2)$</u>.

(a) Show that,

$$\mathbf{H}^* = \mathbf{H} + \mathcal{O}(\epsilon^2) \quad \Rightarrow \mathbf{E}^* = \mathbf{E} + \mathcal{O}(\epsilon^2) \tag{10}$$

Hint: To show the first part, use (7b) and note $\mathbf{F}^{-1} = (\mathbf{I} + \mathbf{H})^{-1} = \mathbf{I} - \mathbf{H} + \mathcal{O}(\epsilon^2)$. The second part immediately follows.

(b) Show that $\lambda_i - 1 = \mathcal{O}(\epsilon)$ where λ_i are eigenvalues of $\mathbf{U}$ (and $\mathbf{V}$). You can use $\mathbf{U} = \mathbf{I} + \mathbf{E} + \mathcal{O}(\epsilon^2)$ (the last relation can be relatively easily shown, but you do not need to prove it). Then, use the relation $\mathbf{U}\mathbf{u}_i = (\mathbf{I} + \mathbf{E} + \mathcal{O}(\epsilon^2))\mathbf{u}_i = \lambda_{\underline{i}}\mathbf{u}_i$ (no summation on i).

(c) Use Taylor expansion of $e(\lambda)$ around 1: $e(\lambda) = e(\lambda - 1 + 1) = e(1) + (\lambda - 1)e'(1) + \frac{(\lambda-1)^2}{2}e''(1) +$ H.O.T. and (2), (3) to show that,

$$\epsilon_e = e(\mathbf{U}) = \mathbf{U} - \mathbf{I} + \mathcal{O}(\epsilon^2) = \mathbf{E} + \mathcal{O}(\epsilon^2) = \mathbf{E}^* + \mathcal{O}(\epsilon^2) \tag{11a}$$
$$\epsilon_e^* = e(\mathbf{V}) = \mathbf{V} - \mathbf{I} + \mathcal{O}(\epsilon^2) = \mathbf{E}^* + \mathcal{O}(\epsilon^2) = \mathbf{E} + \mathcal{O}(\epsilon^2) \tag{11b}$$

You do not need to show (11b) and only (11a) suffices. The proof for the former follows exactly the same line as done in this part and previous parts of this question for $\mathbf{E}, \mathbf{H}, \mathbf{U}$.
Note: We observe that all strain measures satisfying (2), (3) are equal to $\mathbf{E} + \mathcal{O}(\epsilon^2) = \mathbf{E}^* + \mathcal{O}(\epsilon^2)$. So, under infinitesimal deformation gradient conditions all strain measures are equivalent to within $\mathcal{O}(\epsilon^2)$. In addition, for any strain e we have $\varepsilon_e(\mathbf{x}, \mathbf{e}_x) = \varepsilon_e^*(\mathbf{x}, \mathbf{e}_x) = \left(\frac{|\mathrm{d}\mathbf{y}| - |\mathrm{d}\mathbf{x}|}{|\mathrm{d}\mathbf{x}|}\right)$ (*i.e.*, our initial definition of Lagrangian length-based strain) + $\mathcal{O}(\epsilon^2)$; for an example, it is easy to observe that $\left(\frac{|\mathrm{d}\mathbf{y}| + |\mathrm{d}\mathbf{x}|}{2|\mathrm{d}\mathbf{x}|}\right) = 1 + \mathcal{O}(\epsilon)$ in (5). One can also show that ϵ_{eij} and ϵ^*_{eij}, $i \neq j$ is half of the angle change between orthogonal directions $\mathbf{e}_i$, $\mathbf{e}_j$ again to within $\mathcal{O}(\epsilon^2)$.

(d) Infinitesimal change of volume / Volumetric strain: Show that,

$$J = \det \mathbf{F} = 1 + \mathrm{trace}(\mathbf{H}) + \mathcal{O}(\epsilon^2) = 1 + \mathrm{trace}(\mathbf{E}) + \mathcal{O}(\epsilon^2) \quad \Rightarrow$$
$$\epsilon_v := \frac{\mathrm{d}V_y - \mathrm{d}V_x}{\mathrm{d}V_x} = \mathrm{trace}(\mathbf{E}) + \mathcal{O}(\epsilon^2) \tag{12}$$

(e) Infinitesimal change of area. We have $\mathrm{d}\mathbf{A}_y = J\mathbf{F}^{-\mathrm{T}}\mathrm{d}\mathbf{A}_x$. Show that under infinitesimal deformation,

$$\mathrm{d}\mathbf{A}_y = J\mathbf{F}^{-\mathrm{T}}\mathrm{d}\mathbf{A}_x = \mathrm{d}\mathbf{A}_x + \left[(\mathrm{trace}\mathbf{E})\mathbf{I} - \mathbf{H}^{\mathrm{T}}\right]\mathrm{d}\mathbf{A}_x + +\mathcal{O}(\epsilon^2) \tag{13}$$

3. (**20 Points**) Exercise 65 and in addition show that $\gamma_{ij} = C_{ij} + \mathcal{O}(\epsilon^2) = 2E_{ij} + \mathcal{O}(\epsilon^2)$ $(i \neq j)$.

4. (**30 Points**) Exercise 76.

5. (**20 Points**) Show that,

$$\frac{\mathrm{D}J}{\mathrm{D}t} := \dot{J} = J\mathrm{div}\mathbf{v}, \quad \text{where} \quad \mathrm{div}\mathbf{v} = \mathrm{trace}(\nabla_y \mathbf{v}). \tag{14}$$

Hint: Use (15) for $\mathbf{A} = \mathbf{F}$ and $\alpha = t$. Note that this identity was proven during the class.

Exercise problems for your practice (DO NOT need to return them)

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1. Exercise 75. Also show that the given strain field is compatible. For the use of cases where we need to derive displacement from strain field and applications of Airy stress functions refer to resources provided at the course webpage.

2. Show,

$$\frac{\mathrm{d}(\det \mathbf{A})}{\mathrm{d}\alpha} = \mathrm{trace}\left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}\alpha}\mathbf{A}^{-1}\right)\det \mathbf{A} \quad \alpha \text{ any argument (dependency) of } \mathbf{A} \text{ such as time } t \tag{15}$$

Hint: In Abeyaratne Vol I. Exercise 3.7 (page 65) a novel proof of this equation is provided. You can fill out the missing parts in the proof (if any).

3. Different strain measures: Saouma Example 4-7 (page 88).
4. Relation between $\mathbf{x}$ and $\mathbf{y}$: Saouma Example 4-2 (page 78).

Reading assignment: Detailed exposure to topics discussed or those not covered in the class

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- **Strain**: Abeyatratne Vol II sections (2.6), 2.7, 2.8; Saouma sections 4.2.3, 4.2.4 (particularly 4.2.3.1.2, 4.2.4.2.1, 4.2.3.2.2, Table 4.1 and equation 4.2), 4.2.5.
- **Rates of changes of length, angle and area**: You can refer to any of the recommended textbooks to read more about this topic. Sections "3.3 Velocity Gradient, Stretching and Spin Tensors." and "3.4 Rate of Change of Length, Orientation, and Volume" from Abeyaratne Vol II are good references for this topic.
  - Review "Worked Examples": Problems 3.4, 3.5, and 3.7 in that section.