

1. Figure 1 shows a possible jump manifold Γ in a domain \mathcal{D} . The balance law,

$$\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \, dS + \int_{\Omega} \mathbf{r} \, dV = \mathbf{0} \quad (1)$$

holds for all $\Omega \subset \mathcal{D}$. The flux tensor is denoted by \mathbf{F} and \mathbf{r} is the source term. Show that a jump condition across Γ satisfies the condition

$$\mathbf{F}^+ \cdot \mathbf{n}^+ = -\mathbf{F}^- \cdot \mathbf{n}^- \quad \text{which is equivalent to} \quad (2a)$$

$$[[\mathbf{F}]]|_{\Gamma} \cdot \mathbf{n}^+ = (\mathbf{F}^+ - \mathbf{F}^-) \cdot \mathbf{n}^+ = \mathbf{0} \quad (2b)$$

for all $\mathbf{x}_0 \in \Gamma$. We assume that the flux \mathbf{F} is continuous on both sides of Γ .¹ (40 points)

Hint: To prove this equation use the balance law for three domains: Ω^+ , Ω^- , and $\Omega = \Omega^+ \cup \Omega^-$. By subtracting the balance equations on Ω^+ and Ω^- from that on Ω obtain an integral equation solely on $\Gamma \cap \partial\Omega^+$. Then, “localize” the integral to an arbitrary point \mathbf{x}_0 to obtain the point-wise condition.

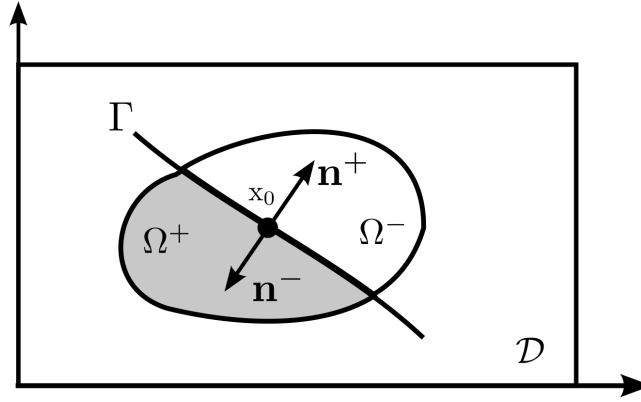


Figure 1: Derivation of the jump conditions from the balance law

2. For elastostatics \mathbf{F} in equation (1) is $-\sigma$ and $\mathbf{r} = \rho\mathbf{b}$, where σ is the stress tensor and \mathbf{b} is the body force per unit mass. (40 points)

- (a) Show that the jump condition for elastostatics across a jump manifold (equation (2a)) reduces to

$$\mathbf{t}^+ = -\mathbf{t}^- \quad (\text{Newton's second law for tractions}). \quad (3)$$

Hint: Note that traction is given by $\mathbf{t} = \sigma \cdot \mathbf{n}$ where for traction on each side, σ and \mathbf{n} are taken from the same side.

- (b) Figure 2 shows an example of jump in flux tensor $-\sigma$ for elastostatics. The top surface is uniformly moved up to generate a uniform strain $\epsilon_{22} = \bar{\epsilon}$ in both materials. Since they possess different elastic moduli E^+ and E^- , the stress tensor has a jump across the material interface Γ . Using the solutions provided in the figure, show that in spite of this jump the tractions still satisfy the common action-reaction law at the interface (*cf.* equation (3)).

- (c) For this 2D problem, the strong form is:

$$\sigma_{11,1} + \sigma_{12,2} + \rho b_1 = 0$$

$$\sigma_{21,1} + \sigma_{22,2} + \rho b_2 = 0$$

¹While physically jumps can occur in \mathbf{F} , the strong form of the problem, *i.e.*, $\nabla \cdot \mathbf{F} - \mathbf{r} = 0$, may not be valid at the points on the jump manifold as the (partial) derivatives in \mathbf{F} may not exist.

Discuss whether any of the partial derivatives in this equation cannot be computed across the jump, given that the stress tensor is discontinuous.

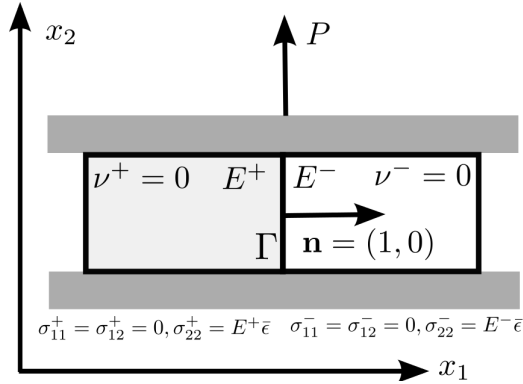


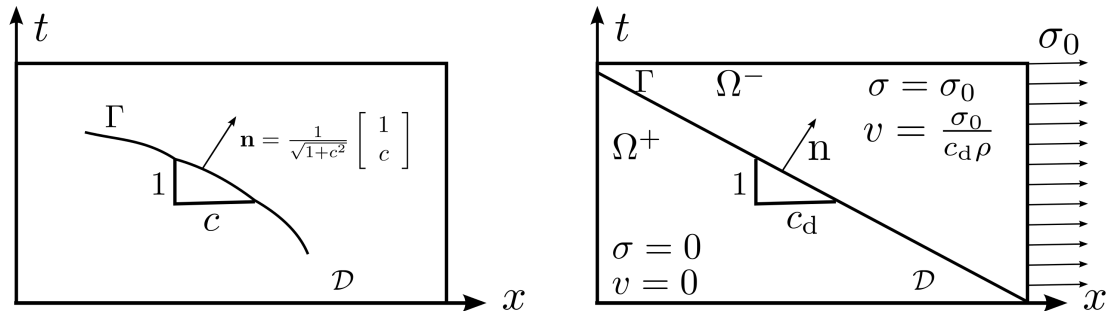
Figure 2: Jump in stress tensor across a material interface.

3. Figure 3(a) shows a possible jump manifold in spacetime for 1D setting. In this case the stress matrix and linear momentum $\mathbf{p} = \rho\mathbf{v}$ reduce to only one component of σ_{11} and p_1 , respectively. For brevity, we drop the these indices. As mentioned, the flux matrix for elastodynamics in spacetime is $\mathbf{F} = [-\sigma \ p]$. (40 points)

- (a) Given the equation (2b), write the jump conditions for this problem based on the solutions and spacetime normal vector ² given in figure 3(a).
- (b) Figure 3(b) shows the exact solution to a 1D wave propagation problem where the right boundary of a semi-infinite bar is loaded by stress σ_0 . Here $c_d = \sqrt{E/\rho}$ is the material dilatational wave speed. Verity that the jump from the two sides + and - satisfy the jump condition obtained in previous step.
- (c) The strong form of this problem is,

$$\sigma_{,x} + \rho b = \dot{p}$$

Given the exact solution in the figure, discuss the validity of the strong form for the points on Γ . Compare this with the elastostatics case in problem 2.c.



(a) Spacetime normal vector \mathbf{n} for a jump manifold moving with speed $-c$. (b) Exact solution for a problem with applied stress

Figure 3: Jump conditions for elastodynamics

²As can be seen, in the calculation of the normal vector, we are adding the square of the speed of the manifold Γ , c^2 , to unity. While the problem can be fixed by using more advanced notations, we do not pursue it further in this course.