Parabolic PDEs:

Homework 4: Part 1:

- Complete `ComputeDG_KM` and `ComputeDG_F` functions in `PDE1DFEM` class and make sure what is computed for **K** and **F** works for parabolic cases as well (if not, make any necessary parabolic changes / implementation needed). Also, add / complete implementation for computing **M** in `ComputeDG_KM` for parabolic PDE.
- Complete the time marching scheme in `function [objout, F, A, slnDGXs, slnDGYs, slnXs, slnYs] = ComputeDG_Sln_Parabolic(obj)` by using forward Euler method.
- Then run "config_DG_PeriodicSine.txt" and send me your `PDE1DFEM.m` file along with two generated files `config_DG_PeriodicSine_u.png` and `config_DG_PeriodicSine_DuDx.png` and the output file `config_DG_PeriodicSine.out` after `scriptSolvePDE.m` is ran with `configNameWOExt = 'config_DG_PeriodicSine';` is chosen as the active config file (last entry). Note that no other file must be changed.

Homework 4: Part 2 (Be **brief** in your answers. Try to answer all questions in less than two pages (but be specific and to the point answering the question asked).

1. Stability limit, we need to find a stability limit factor f, so that

$$\Delta t = f \min_e \frac{c_e h_e^2}{2\kappa_e} \tag{1}$$

so that forward Euler scheme is stable for

$\epsilon = 0$, $\sigma = 0$, and element polynomial $p = 1$. Other cases can be studies similarly (different parameters for star value, different element order and time integration scheme). Use `scriptSolvePDE.m` file to run the following cases (configNameWOExt can be changed in the beginning of the file). Based on your observation comment on the maximum value of f that can be used for this set-up preferably to 3 digits.

configNameWOExt	f	`numberOfElements`
`config_DG_LeftEssential0_RightNatural1`	1	8
`config_DG_PeriodicSine`	0.5	8
`config_DG_LeftEssential0_RightNatural1`	0.5	8
`config_DG_PeriodicSine`	0.35	8
`config_DG_LeftEssential0_RightNatural1`	0.35	8
`config_DG_LeftEssential0_RightNatural1`	0.34	8
`config_DG_PeriodicSine`	0.334	8
`config_DG_LeftEssential0_RightNatural1`	0.334	8
`config_DG_PeriodicSine`	0.334	64
`config_DG_LeftEssential0_RightNatural1`	0.334	64
`config_DG_PeriodicSine`	0.334	128
`config_DG_PeriodicSine`	0.334	256
`config_DG_LeftEssential0_RightNatural1`	0.333	64
`config_DG_PeriodicSine`	0.333	256

Table 1: A sample study on stability factor for a DG parabolic solver in 1D.

2. Run `MAIN_PDEComparison.m` with `config_DG_LeftEssential0_RightNatural1.txt` and `config_DG_PeriodicSine.txt` as the active config entries (ensure that PDEmode is 1). Comment on the following:

   (a) What can be said for the choice $\epsilon = -1$ (for $\sigma = 0$ or $\sigma = 1$)?

   (b) Again for $\epsilon = -1$, run `scriptSolvePDE.m` for $\epsilon = -1$ and $\sigma = 10$, `configNameWOExt` = `config_DG_LeftEssential0_RightNatural1`, `numberOfElements` = 8, `stabilityDelTFactor` = 0.0001. What can be said on the stabilty of scheme for $\epsilon = -1$? What can be said about the symmetry of $\mathbf{K}$?

   (c) $\epsilon = 0$, $\sigma = 0$:

      i. What would this scheme be called from an interior penalty perspective (refer to "Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations", by Riviere)?

ii. Refer to "Shu_2001_Different formulations of the discontinuous Galerkin method for the viscous terms.pdf" paper shared with you (Referred to Shu (2001) from here on. There are three schemes discussed under item numbers 3. 4. (LDG), 5. (Baumann-Oden formulation). The choice $\epsilon = 0$, $\sigma = 0$ corresponds to which of these three schemes?

iii. Based on the discussion in Shu (2001) and your results for both periodic sine wave and $q = 1$ on the left hand side, what can be said about the stability, consistency, and converge of the scheme for $\epsilon = 0$, $\sigma = 0$? (Be brief, less than 3-4 lines of explanation). If the scheme is nonconvergent, explain what may contribute to this response from a weighted residual formulation perspective (respond in 1 short sentence).

(d) $\epsilon = 0$, $\sigma = 1$: What is the effect of adding (sufficiently large) $\sigma \neq 0$ to the formulation?

(e) $\epsilon = 1$:

i. What can be said about the performance of the method for this choice?

ii. What is this scheme called from an interior penalty formulation perspective (consider both cases $\sigma = 0$ and $\sigma \neq 0$)? Just mention the methods names.

iii. Referring to Shu (2001), which scheme does this correspond to (among the 3 presented)? What are its orders of accuracy for odd and even element interpolation order p (referred to as k in that paper)?

iv. What is the lowest element order p (k) that can be chosen for this scheme? Explain in less than 2 lines why lower order polynomials cannot be used?

v. Is $\mathbf{K}$ symmetric (yes / no)?

vi. Based on the discussion in Shu (2001), which one of the three schemes under numbers 3., 4., and 5. would you use? Explain you answer in less than 3 lines in terms of order of accuracy, simplicity of the flux, and element orders that can be used.

3. Other choices for parabolic fluxes: Unlike hyperbolic PDEs where Riemann solutions exist and can be obtained by taking the upwind value of characteristic waves, such cannot be done for parabolic (and elliptic) PDEs. The choice of jump terms added to average fluxes requires rigorous analysis to understand their effect on stability and performance of the method. There are, however, some attempts in the literature to obtain spatial flux contributions at element edges. The paper "Lorcher_2008_An explicit discontinuous Galerkin scheme with local time-stepping for general unsteady diffusion equations" provides some insight in this process. Equations (2.3) to (2.5) provide the PDE studies (so μ stands for κ/c in our formulation; equation (3b) in previous assignment). Please follow the solution process in Appendix A on how the average spatial flux from time t_n to $t_n + \Delta t$ is carried. The final result is one of the equations (A.6) to (A.8). Notice, that point-wise solution at the interface right at $t - t_n = 0$ is singular, and only time-integral of flux over Δt span makes sense. Having found the right equation, compute time-integral of spatial flux for the following parameters:

$$
\begin{aligned}
&\mu^- = 1 && \mu^+ = 3 \\
&u^- = 1.5 && u^+ = 2.5 \\
&f_{\vec{n}}^- = \mu^- u^-, x = 2 && f_{\vec{n}}^- = \mu^+ u^+, x = 4 \\
&\Delta t = 0.3 &&
\end{aligned}
$$