The balance of energy for purely thermal response is,

$$C\dot{T} + \nabla.\mathbf{q} = Q \tag{1}$$

in which $C = \rho c_p$ is the volumetric heat capacity where ρ is the mass density and c_p is is the specific heat capacity, T is temperature, $\mathbf{q}$ is heat flux vector, and Q is volumetric heat source. The MaxwellCattaneoVernotte (MCV) modification to heat flux equation is

$$\tau\dot{\mathbf{q}} + \mathbf{q} = -\boldsymbol{\kappa}\nabla T \tag{2}$$

where the relaxation time τ is added to Fourier heat flux equation $\mathbf{q} = -\boldsymbol{\kappa}\nabla T$ with $\boldsymbol{\kappa}$ being thermal conductivity matrix.

1. (**200 Points**) Conservation law representation of hyperbolic heat equation:

   (a) Show that (1) and (2) yield the second order PDE,

   $$\tau C\ddot{T} + C\dot{T} - \nabla.(\boldsymbol{\kappa}\nabla T) = Q + \tau\dot{Q} \tag{3}$$

   (b) To evaluate whether the equation is hyperbolic or not, we need to examine the possibility of wave motion in arbitrary direction $\mathbf{n}$ in space in the form $T = \bar{T}(\mathbf{n}.\mathbf{x} - ct)$ with c being the wave speed. To simplify, we simplify the problem to 1D and use homogeneous material properties. This 1D hyperbolicity analysis carries to 2D and 3D if the material is isotropic (*i.e.,* $\boldsymbol{\kappa}$ is diagonal $\boldsymbol{\kappa} = k\mathbf{1}$). The 1D equation for homogeneous material is,

   $$\tau C\ddot{T} + C\dot{T} - k\mathit{\Delta} T = Q + \tau\dot{Q} \tag{4}$$

   Show that (4) equation is hyperbolic.

   (c) Show that the equations (1) and (2) can be written in the form of system of conservation laws,

   $$\dot{\mathbf{U}} + \mathbf{A}\mathbf{U}_{,x} = \mathbf{S}, \quad \mathbf{U} = \begin{bmatrix} CT \\ \tau q \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & \frac{1}{\tau} \\ \frac{k}{C} & 0 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} Q \\ -q \end{bmatrix} \tag{5}$$

   Note that q is expressed as a scalar in 1D.

   (d) What are the conditions for the hyperbolicity of (5), in terms of the flux matrix $\mathbf{A}$. Show that (5) is in fact hyperbolic.

2. (**300 Points**) SDG solution for the thermal problem: In the class we solved patch p_1 for the following initial boundary value problem:

Domain	$[0\ 3] \times \mathbb{R}$
Material properties:	$C = 1, k = 1, \tau = 1, \quad \Rightarrow \quad c = 1$
Initial Conditions:	$T_0(x, 0) = 0, q_0(x, 0) = 0$
Boundary Conditions:	$\bar{T}(0, t) = 0$ (Dirichlet BC) , $\bar{q}(3, t) = 1$ (Neumann BC)

   Using the solution we obtained for element e_1 in patch p_1,

   $$T^1(\underline{x}, \underline{t}) = -3.27\underline{t} \tag{6a}$$

   $$q^1(\underline{x}, \underline{t}) = -6 + 3.27\underline{x} + 15.27\underline{t} \tag{6b}$$

   $$\underline{x} = x - 2, \underline{t} = t \quad \text{local Cartesian coordinate for element } e_1 \tag{6c}$$

   answer the following questions:

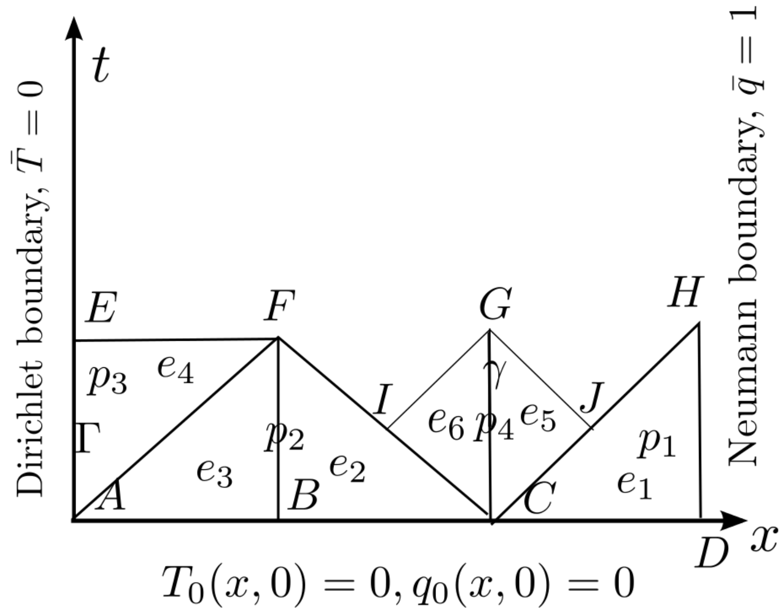

Figure 1: Schematic of SDG mesh, and initial and boundary conditions.

Table 1: Vertex coordinates for the mesh shown in figure 1

Vertex	Coordinate
A	(0, 0)
B	(1, 0)
C	(2, 0)
D	(3, 0)
E	(0, 1)
F	(1, 1)
G	(2, 1)
H	(3, 1)
I	(1.5, 0.5)
J	(2.5, 0.5)

(a) Can the patch p_2 be solved in parallel to patch p_1? What is patch the solution for p_2?

(b) What patches must be solved before patch p_3 can be solved? What is the solution for p_3? What are T^* and q^* for the the facet of e_4 on the spatial boundary of the domain (Γ)?

(c) What polynomial order is used for the solution of the element(s) in patch p_1?

(d) Patch p_4 is comprised of elements e_5 and e_6. In terms of hp (h element size, p polynomial order) of the element briefly (less than 2-4 sentences for each question) answer the following question.

i. Can the element e_6 have a different polynomial order than e_1 without using any transition elements? How is this different from continuous FEM (CFEM)?

ii. Can we use different polynomial orders for T and q interpolation in element e_6?

iii. Can elements e_5 and e_6 have different polynomial order?

iv. Explain why the nonconforming geometry transition between elements e_1 and e_6 and between elements e_2 and e_5 can be easily handles in SDG method. Explain how nonconforming h transition in SDG method is different from CFEM method.

(e) Why elements e_5 and e_6 should be solved simultaneously?

(f) For the solution of p_4 use 1st order polynomial interpolation for temperature field T of element e_6 and constant interpolations for T of element e_5 and q field of both elements:

$$T^5(\underline{x}, \underline{t}) = a_1 + a_2\underline{x} + a_3\underline{t} \tag{7a}$$

$$q^5(\underline{x}, \underline{t}) = a_4 \tag{7b}$$

$$T^6(\underline{x}, \underline{t}) = a_5 \tag{7c}$$

$$q^6(\underline{x}, \underline{t}) = a_6 \tag{7d}$$

where $(\underline{x}, \underline{t})$ are the same local coordinate used for the solution of element e_1:

$$\underline{x} = x - 2, \underline{t} = t \tag{8}$$

Use average flux for the target values on vertical facet γ

$$T^*(\underline{x}, \underline{t}) = \frac{1}{2}\left(T^5(\underline{x}, \underline{t}) + T^6(\underline{x}, \underline{t})\right) \tag{9a}$$

$$q^*(\underline{x}, \underline{t}) = \frac{1}{2}\left(q^5(\underline{x}, \underline{t}) + T^6(\underline{x}, \underline{t})\right) \tag{9b}$$

and solve for the unknowns a_1 to a_6.