Riemann solutions for MCV equation.

1. (**200 Points**) Riemann solution for same material properties on the two sides: Using left eigenvalues and eigenvectors of $\mathbf{A}$ present $\mathbf{U} = [T\ q]^{\mathrm{T}}$ in regions R^I, R^{II}, R^{III} in figure 2.

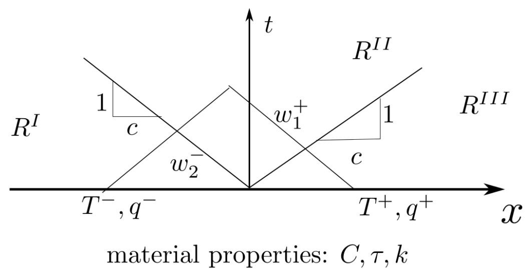
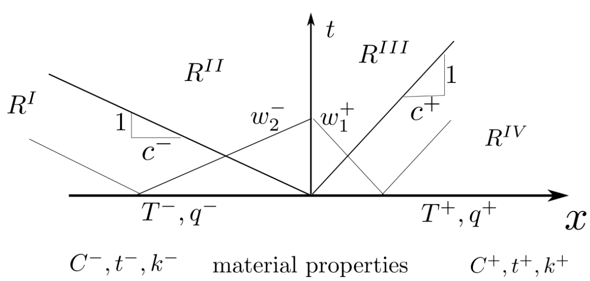
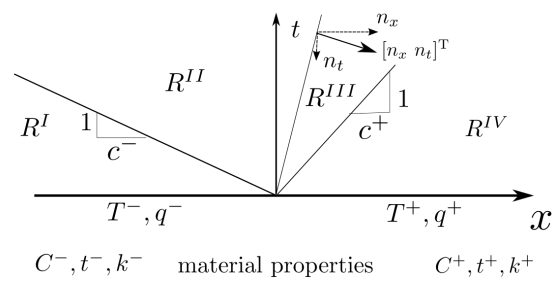


Figure 2: Initial conditions, solution regions, and wave speeds for Riemann problem of 1D MCV heat equation with same material properties.



(a) Riemann solution using characteristics method.



(b) Riemann solution using allowable jump conditions.

Figure 3: Riemann problem for MCV heat equation, different material properties: Two different solution methods

2. (**150 Points**) Riemann solution with different material properties on the two sides of the interface: Referring to figure 3(a) extend your previous solution to an interface with jump in material properties. For simplicity adopt the following definition,

$$Z := cC = \sqrt{\frac{kC}{\tau}} \tag{10}$$

Note that there will be different solutions in regions R^{II} and R^{III} due to abrupt change in material properties. The balance of energy equation (1) implies that $\mathbf{q}$ must remain continuous across the vertical material boundary. In addition, we may stipulate that corresponding to (2) temperature T remains continuous across vertical boundary. Thus, while $[\![q]\!] = 0, [\![T]\!] = 0$ across the vertical boundary, other relevant fields such as conserved quantities have nonzero jumps: $[\![CT]\!] \neq 0$, $[\![\tau q]\!] \neq 0$. Show that the Riemann solution in regions II and III are,

$$T = \frac{1}{Z^- + Z^+}\left\{(T^-Z^- + T^+Z^+) - (q^+ - q^-)\right\} \tag{11a}$$

$$q = \frac{1}{Z^- + Z^+}\left\{-Z^-Z^+(T^+ - T^-) + (Z^-q^+ + Z^+q^-)\right\} \tag{11b}$$

3. (**150 Points**) Riemann solution using allowable jump condition: For a conservation law in the form,

$$\mathbf{f}_{t,t} + \nabla.\mathbf{f}_x = \mathbf{S} \tag{12}$$

we observed that the jump condition was in the form,

$$[\![\mathbf{F}]\!].\mathbf{N} = 0 \quad \Rightarrow \quad [\![\mathbf{f}_t]\!]n_t + [\![\mathbf{f}_x]\!]n_x = 0 \tag{13}$$

Noting that for the 1D conservation law (5), $\mathbf{f}_t = \mathbf{U}, \mathbf{f}_x = \mathbf{AU}$, and $\mathbf{n}_x = n_x$ (scalar normal vector and only x component spatial flux for 1D) we obtain,

$$[\![\mathbf{F}]\!].\mathbf{N} = 0 \quad \Rightarrow \quad [\![\mathbf{f}_t]\!]n_t + [\![\mathbf{f}_x]\!]n_x = 0 \tag{14}$$

Finally noting that for a given ray with speed c, $c = -n_x/n_t$ we obtain (*cf.* figure 3(b)),

$$\begin{aligned} \{[\![\mathbf{f}_t]\!] - c[\![\mathbf{f}_x]\!]\}\, n_t &= 0 \\ \mathbf{f}_t &= \mathbf{U} \qquad \Rightarrow \quad [\![\mathbf{AU}]\!] = c[\![\mathbf{U}]\!] \\ \mathbf{f}_x &= \mathbf{AU} \end{aligned} \tag{15}$$

when the jump condition is in one of the two domains $[\![\mathbf{AU}]\!] = \mathbf{A}[\![\mathbf{U}]\!]$. That is,

$$[\![\mathbf{AU}]\!] = c[\![\mathbf{U}]\!] \quad \Rightarrow \mathbf{A}[\![\mathbf{U}]\!] = c[\![\mathbf{U}]\!] \Rightarrow \tag{16}$$

$$\boxed{c \text{ and } [\![\mathbf{U}]\!] \text{ are right eigenvalue and eigenvectors of } \mathbf{A}, \text{ for } c \neq 0} \tag{17}$$

Finally we need to satisfy jump condition on the material interface employing equation (15),

$$[\![\mathbf{AU}]\!] = c[\![\mathbf{U}]\!], \quad c = 0, \quad \Rightarrow \boxed{\mathbf{A}^+\mathbf{U}^+ = \mathbf{A}^-\mathbf{U}^-, \quad c = 0 \text{ (material interface)}} \tag{18}$$

As explained in the class, by successively adding the jump conditions from − to + side we obtain a system of equation in terms of the magnitude of jumps across each characteristic direction. A trivial way to ensure the satisfaction of (18) is using primary fields T and q as the components of $\mathbf{U}$.

Again by using the convention $Z = cC$ (*cf.* (10)) derive right eigenvalues and eigenvectors of A (A^+, A^-), and solve for magnitudes of jumps across $-c^-$ and c^+ directions α^-, α^+, respectively. Finally demonstrate that the solution from this method matches that from characteristic values approach in equation (11).