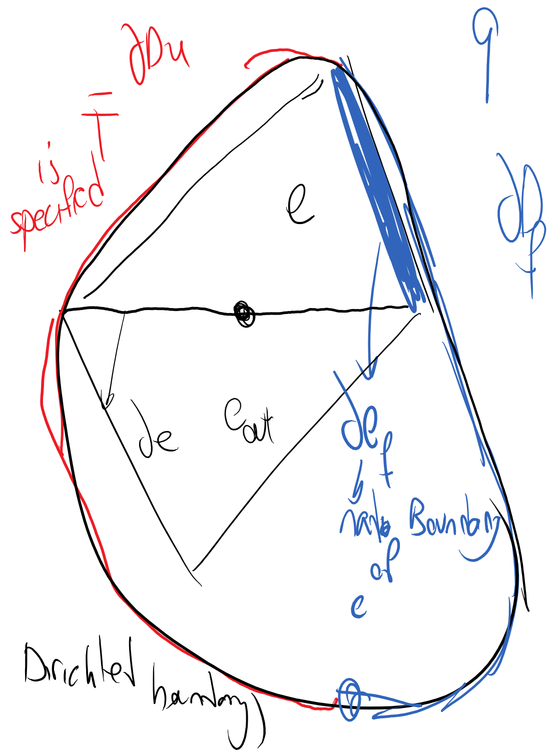
2018/01/29

Monday, January 29, 2018 11:42 AM

$R_u = \bar{T} - T$



changes relative to CFEMs

① R_u is specified ONLY

on ∂D_u (Domain Drichlet boundary)

in FEM's strongly

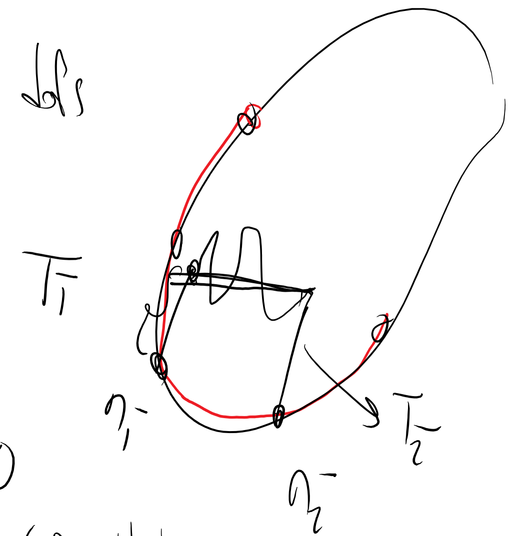
& inside the domain again it's strongly satisfied by using nodal dofs

$$T = \underbrace{\sum_{i=1}^{n_f} a_i^f N_i^f}_{\text{takes care of continuity } (T^+ - T^- = 0) \text{ inside the domain}} + \underbrace{\sum_{i=1}^{n_p} a_i^p N_i^p}_{\text{takes care of } R_u = 0 \text{ at nodes on } \partial D_u \text{ (prescribed nodes)}}$$

So CFEM satisfies $R_u=0$ strongly

side note ①

$R_u \neq 0$ between prescribed dofs



side note ②

Compare satisfaction of ⓐ $\underbrace{[T]}_{\text{Jump in } T} = 0$ (Dirichlet type Jump)

versus ⓑ $[q] \cdot n = 0$
Neumann type Jump

in CFEMs.

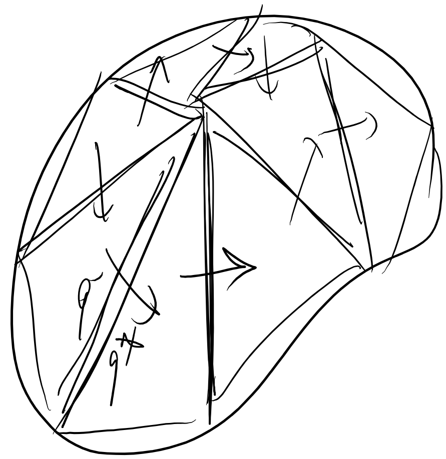
Obviously ⓐ is satisfied strongly by construction
But ⓑ is not <u>explicitly</u> enforced

Idea: What if we penalize jumps

in (b) even in CFEMs...

NR = Common NR +

$$P \sum_{\Gamma} \int (q^+ - q^-) \cdot n \, ds = 0$$



P: large number

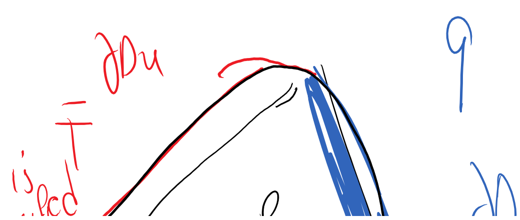
Γ: the union of interior faces to the domain

q^+, q^- heat fluxes from the two sides of the interface

This idea of interior penalty can be viewed as another way to formulate DG methods (as opposed to numerical flux idea)

Going back to DG treatment of essential type jumps …

$$R_u = \bar{T} - T$$



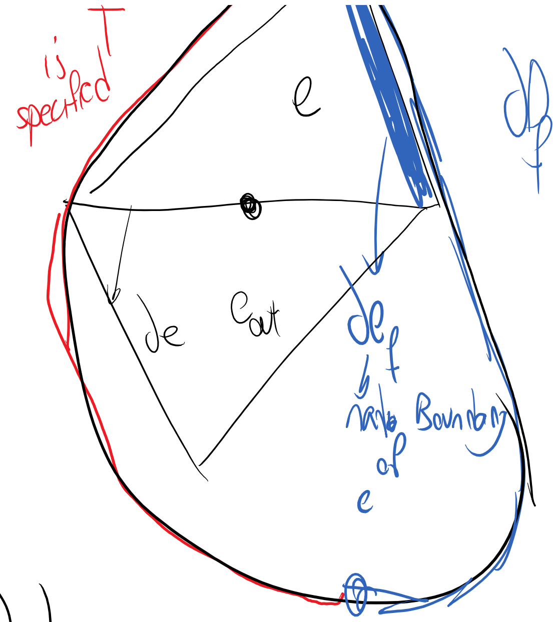
$R_u = T - \bar{T}$

is enforced weakly

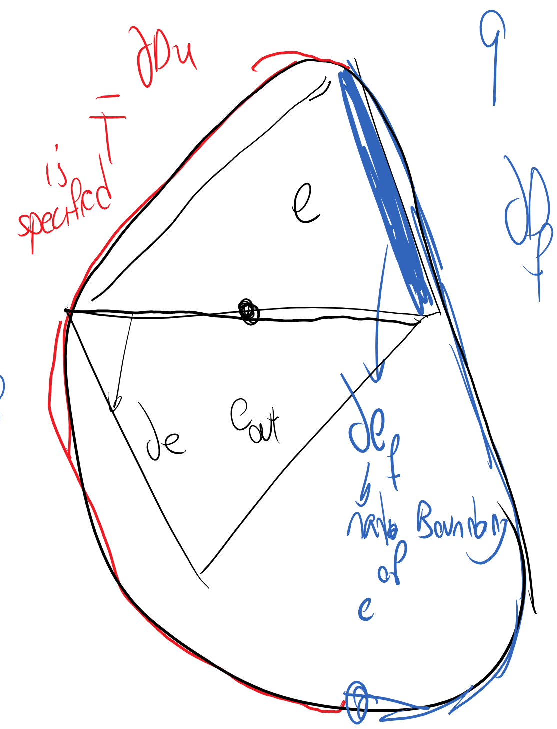
& as R_f it's enforced across all element boundary

to do this & extend $\bar{T}(\partial D_u)$ to all element boundaries we use T^* notation

T^* : Numerical "flux" for T



$$T^* = \begin{cases} \bar{T} & \text{for } \partial e \cap \partial D_u \\ T \text{ (interior trace)} & \text{on } \partial e \cap \partial D_f = \partial e_f \\ & \text{(so } T^* - T = 0 \text{ on } \partial e_f) \\ T^*(T^-, T^+, \cancel{q^-, q^+}) & \end{cases}$$



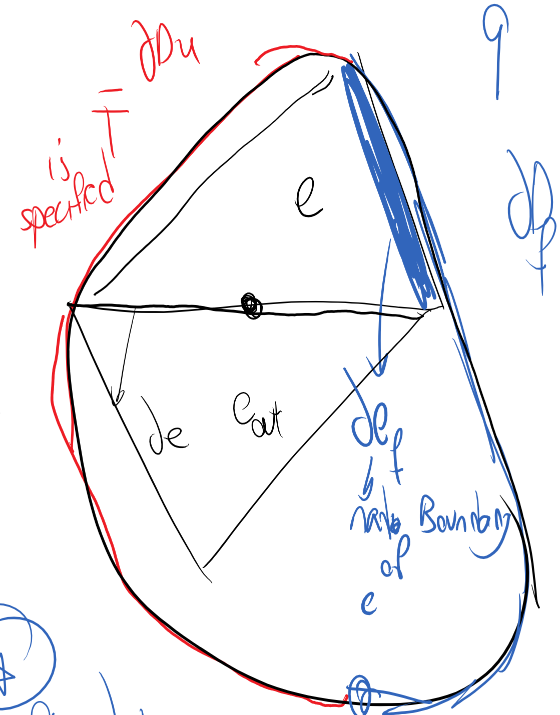
typically $T^- + T^+$

typically $= \frac{T^- + T^+}{2}$

Now that we have all the residuals we can formulate the DG weak form

Find the solution in element e:

$\forall w \in V_h$ → the space of interpolation functions inside the element



$$\int_e w \underbrace{\left(C\dot{T} + \nabla \cdot q - Q\right)}_{R_i} dv + \int_{\partial e} w \underbrace{\left(q^{*} - q \cdot n\right)}_{R_f} ds$$

Done only on ∂D_f in CFEMs

$$+ \boxed{\int_{\partial e} f(w) \underbrace{\left(T^{*} - T\right)}_{R_u} ds} = 0$$

entirely new term in DG methods

linear operator on W:

Some comments on f(w):

1. If we use f(w) = w the dimensions of Ru-based and the rest of weighted residual are not the same, so as an element size -> 0 or increases some terms lose their effect (go to zero relatively for example) … so we basically "don't enforce them":

Remedy (1) multiply by a factor that fixes the physical dimension i.e. consistency (often depends on element size)

$$\frac{r}{h} \int_{\partial e} T\left(T^{*} - T\right) ds$$

(2) use a formulation that is dimensionally consistent.

2. f(w) = q(w) = - $k \nabla T$

I'll use this option.

Comparison:

2. Is more favorable for some energy norm type errors But if the element is interpolated with constant (0th order polynomial) function, this error is NOT enforced

We'll choose option two, and not use p = 0 elements

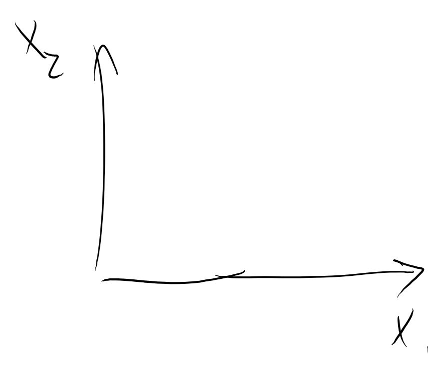
$$\int_e w\cdot\left(c\dot{T} + \nabla\cdot q - Q\right)dv + \int_{\partial e} w\left(q^{*} - q\cdot n\right)ds + \int_{\partial e} \underbrace{\overset{-k\nabla w}{\overbrace{q(w)}}\cdot n}_{\text{scalar}}\left(T^{*} - T\right)ds = 0$$

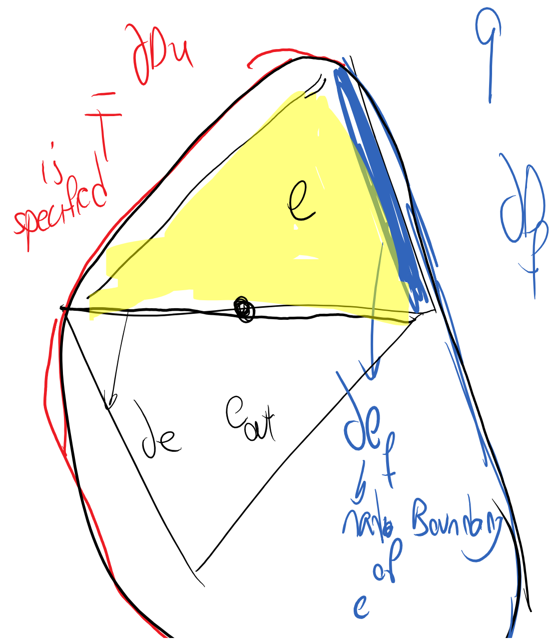
application of Gauss theorem

$$\int_e w\nabla\cdot q\,dv = \int_e -\nabla w\, q\,dv + \int_{\partial e} w q\cdot n\,ds$$

$$\circledast \quad \boxed{\int_e \left(w c\dot{T} - \nabla w\, q - wQ\right)dv + \int_{\partial e} w q^{*}\,ds + \int_{\partial e} k\nabla w\cdot n\left(T^{*} - T\right)ds = 0}$$

DG weak form





$$T = \underset{\sim}{N}\, a$$

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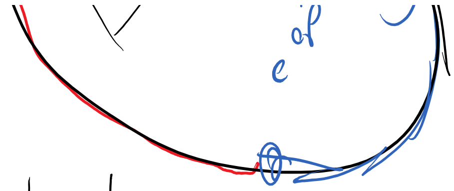
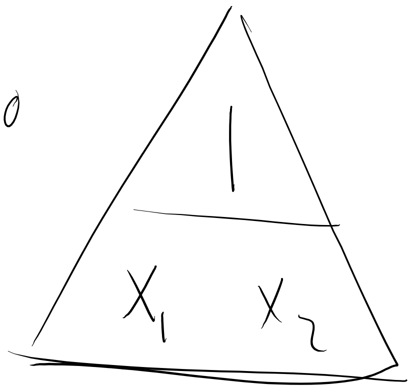


unlike CFEM's don't need to have the delta property $N_i(n_j) \neq \delta_{ij}$ (not needed)

Example p=1 element p=0



p=1

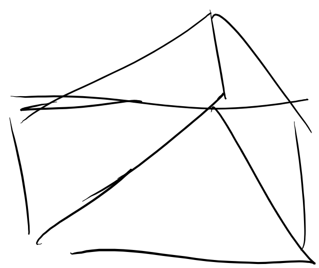
3 terms for p=1

p=2 X_1^2 $X_1 X_2$ X_2^2

⋮

$N = \begin{bmatrix} 1 & x_1 & x_2 \end{bmatrix}$

$\quad N_1 \quad N_2 \quad N_3$



There are no problems with these monomial basis (other than as p increases first we need to use better basis and basis basis coordinate)

How do we solve (*)

$w = N_1, N_2, \ldots$

$w = N^T = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{bmatrix}$

$T \quad N a \quad N(x)\, a(t)$

$$T = \underset{1\times n}{N} a = N(x)\, a(t) \qquad \underset{2\times n}{B} = \nabla N$$

$$\dot{T} = N(x)\, \dot{a}$$

$$\int_e \Big(w C \dot{T} - \underbrace{\nabla w\, q}_{-\nabla w(-k\nabla T)} - wQ \Big)\, dv + \int_{\partial e} w q^* \, ds + \int_{\partial e} k \nabla w \cdot n (T^* - T)\, ds = 0$$

$$\underbrace{R}_{\text{residual vector}} = \int_e \Big[N^T C (N \dot{a}) + B^T k (B a) - N Q \Big]\, dv + \int_{\partial e} N^T q^* \, ds - \int k B \cdot n (T^* - N a)\, ds = 0$$

$$\boxed{R = \Big[\underbrace{\int_e (N^T C N)\, dv}_{me} \Big] \dot{a} + \Big[\int_e (B^T k B)\, dv + \int_{\partial e} k B \cdot n N \, ds \Big] a + \int_e N^T q^* \, ds - \int_{\partial e} k B \cdot n\, T^* \, ds - \int N^T Q = 0}$$

To solve it we need to assemble this system for all elements.

Good design DON'T hard-code the choice of T^*, q^*

$$\begin{bmatrix} T^-, & T^+ \\ q^-, & q^+ \end{bmatrix} \longrightarrow \left| \begin{array}{l} T^*(T^-, T^+, q^-, q^+) \\ \dfrac{\partial T^*}{\partial T^-} \quad \dfrac{\partial T^*}{\partial T^+} \quad \dfrac{\partial T^*}{\partial q^-} \quad \dfrac{\partial T^*}{\partial q^+} \end{array} \right.$$

flux function

same with q^*

1D version and the same 3 element

example

$T = a_1 + a_2 x$ → local dofs $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$T_L = [1 \ 0]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$N = [1 \quad x]$

$T_R = [1 \ L]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$T = [1 \quad x]\begin{bmatrix} a_1 \\ \end{bmatrix}$

R

L $\qquad T = [1 \;\; x]\begin{bmatrix}a_1\\ a_2\end{bmatrix}$ $\qquad$ R

$n_L = -1 \quad x=0 \qquad B = [0 \;\; 1] \qquad x = \ell \qquad n_R = 1$

$q_L = -ka_2 \qquad q = -kBa = -k[0 \;\; 1]\begin{bmatrix}a_1\\ a_2\end{bmatrix} = a_2 \qquad q_R = -ka_2$

$$N = [1 \;\; x] \quad B = \nabla N = \frac{d}{dx}N = [0 \;\; 1]$$

$$\int_e \left(\underbrace{wC\dot{T}}_{m^e} - \nabla w\, q - wQ\right) dV + \int w q^* ds + \int k\nabla w \cdot n (T^* - T)\, ds = 0$$

$$\overbrace{\left(\int_e N^T C N\, dV\right)}^{m^e}\dot{a} + \overbrace{\left(\int_e B^T k B\, dV\right)}^{\tilde{k}^e} a - \overbrace{\int_e N Q}^{fr} + \overbrace{\left(N^T q^*\right)_{x_R} + \left(N q^*\right)_{x_L}}^{\partial e}$$

$$+ \left(-kB^T \cdot n\,(T^* - T)\right)\Big|_{x_R} + \left(kB^T \cdot n\,(T^* - T)\right)_{x_L} = 0$$

$$m^e\dot{a} + \tilde{k}^e a - fr$$

$$+ \begin{bmatrix}1\\ \ell\end{bmatrix} q_R^* + \begin{bmatrix}1\\ 0\end{bmatrix} q_L^*$$

$$-k\begin{bmatrix}0\\1\end{bmatrix}(-1)\left(T_L^{\phi}-\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}\right)$$

$$+k\begin{bmatrix}0\\1\end{bmatrix}(1)\left(T_R^{\phi}-\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}\right)=0$$

$$\boxed{m^e\ddot{a}+\tilde{K}^e a+\begin{bmatrix}q^{\phi L}+q^{\phi R}\\ \ell q^{\phi R}\end{bmatrix}+\begin{bmatrix}0\\ k\left(T^{\phi L}-T^{\phi R}\right)\end{bmatrix}=0}$$