2018/02/05

Monday, February 05, 2018 11:37 AM

DG method

- elliptic & parabolic PDEs
  - Notation of Numerical flux
    - Different choices of (E, P) fluxes
    - Relation to interior penalty method
  - Time marching for parabolic case
- hyperbolic PDEs Normal DG formulation (space + time marching)
  - Star values
    - average flux vs. (Riemann / approximate Riemann)
- Spacetime DG ← 1 project
- Implementation aspects

Project-like assignments

Elliptic, Parabolic, hyperbolic } DG solutions NOT in Spacetime

$$
\begin{array}{c|cc|cc|cc}
\text{weights} \backslash \text{Finally solutions} & e_1 & & e_2 & & e_3 & \\
\hline
e_1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\
 & -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
\hline
e_2 & 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\
 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\
\hline
e_3 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1
\end{array}
$$

$$
K = \begin{bmatrix}
0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1
\end{bmatrix}
\qquad
F = \begin{bmatrix} 0 \\ -T \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}
$$

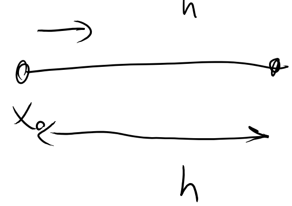
$$M\mathring{a} + Ka = F$$ ①

we got K, F last time

M (Capacity) matrix this time:

$$m^e = \int_e N^T \underset{\text{capacity}}{c}\, N \, \cancel{dX}$$

$$x = \frac{X - X_0}{h}$$



$$\xrightarrow{X}$$

$$= \int \begin{bmatrix} 1 \\ x \end{bmatrix} c \begin{bmatrix} 1 & x \end{bmatrix} \underbrace{(h\,dx)}_{dX}$$

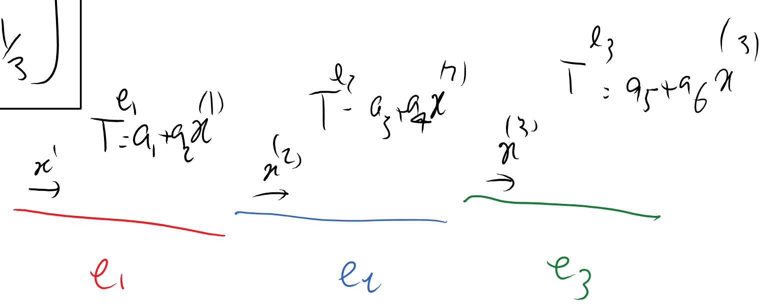
c if constant

$$N = \begin{bmatrix} 1 & x \end{bmatrix}$$

$$x = \frac{X - X_0}{h} \implies dx = \frac{dX}{h}$$

=

$$\boxed{m^e = ch \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}}$$



$c, h = 1$ for all elements

$$m^e = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$ for all elements

to assemble element m^e's to M

by assembling element m^e's to M

$$M = \begin{bmatrix} 1 & \frac{1}{2} & & & & \\ \frac{1}{2} & \frac{1}{3} & & & & \\ & & 1 & \frac{1}{2} & & \\ & & \frac{1}{2} & \frac{1}{3} & & \\ & & & & 1 & \frac{1}{2} \\ & & & & \frac{1}{2} & \frac{1}{3} \end{bmatrix} \qquad (2)$$

$$M\dot{a} + Ka = F$$

with $a = a_0 \rightarrow$ initial condition for parabolic equation

K, F come from eqn (1)

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elliptic problem just needs the solution $Ka = F$

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parabolic problem:

$M\dot{a} + Ka = F$ — Semi-discrete form

x: discretized

t: Not yet

— there are many ways to discretize this in time (Euler, RK, ...)
B.
F.

Implicit time-marching (with backward Euler)

write the equation for the current time step:

$$M \dot{a}_{n+1} + K a_{n+1} = F_{n+1}$$

Using $\quad \dot{a}_{n+1} = \dfrac{a_{n+1} - a_n}{\Delta t}$

we obtain

$$M(a_{n+1} - a_n) + \Delta t K a_{n+1} = \Delta t f_{n+1}$$

$$\rightarrow \quad \boxed{\begin{gathered} \underbrace{(M + \Delta t K)}_{\mathcal{M}_{n+1}} a_{n+1} = \underbrace{\Delta t f_{n+1} + M a_n}_{F_{n+1}} \\ \mathcal{M}_{n+1} a_{n+1} = F_{n+1} \end{gathered}}$$

implicit version

(3a)

Explicit Euler (forward Euler) – write the equation for prev. time step

$$M \dot{a}_n + K a_n = F_n$$

forward FD formula: $\quad \dot{a}_n = \dfrac{a_{n+1} - a_n}{\Delta t}$

we obtain

$$M(a_{n+1} - a_n) + \Delta t K a_n = \Delta t f_n$$

$$\underbrace{M}_{\mathcal{M}} a_{n+1} = \underbrace{(M - \Delta t K) a_n + \Delta t f_n}_{F}$$

explicit version

$$\mathcal{M} a_{n+1} = F$$

$$\boxed{\mathcal{M}_{n+1}\, a_{n+1} = \mathcal{F}_{n+1}}$$ (3b)

$$\underbrace{(M + \Delta t K)}_{\mathcal{M}_{n+1}} a_{n+1} = \underbrace{\Delta t\, f_{n+1} + M a_n}_{\mathcal{F}_{n+1}}$$ (3a) Implict (Bw Euler)

$$\mathcal{M}_{n+1}\, a_{n+1} = \mathcal{F}_{n+1}$$

$$\underbrace{M}_{\mathcal{M}_{n+1}} a_{n+1} = \underbrace{(M - \Delta t K) a_n + \Delta t\, f_n}_{\mathcal{F}_{n+1}}$$ (3b) Explict FW Euler

$$\mathcal{M}_{n+1}\, a_{n+1} = \mathcal{F}_{n+1}$$

Advantages of exp. to imp.

- Exp is always linear (I p. is nonlinear for nonlinear PDEs)
- LHS M changs for
  - nonlinearity
  - Δt changes

  for implicit
- LHS M has a better form in exp
  - CFEM: mass lumping $M \to$ diagonal
  - DG M: block diagonal

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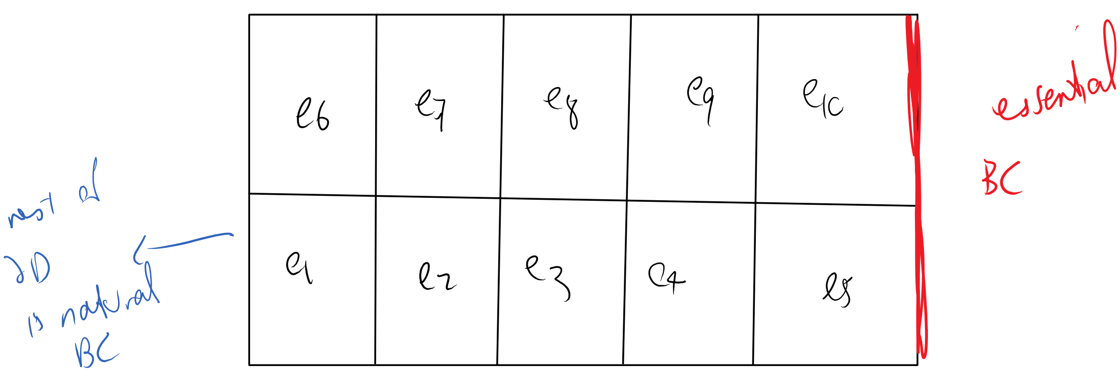
for parabolic problem a_0 is obtained from IC

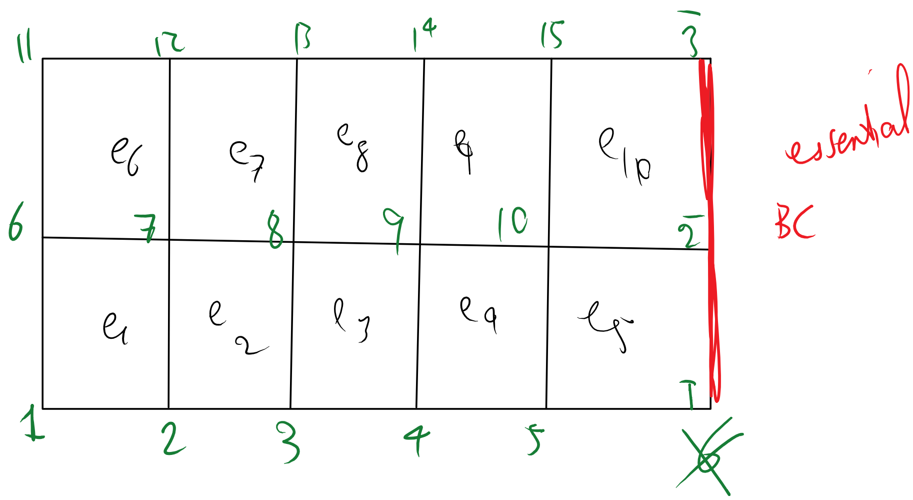
$$M_1 a_1 = F_1(a_0)$$

$$M_2 a_2 = F_2(a_1) \quad \text{in general } F_{n+1}(a_0 \cdots, a_n)$$

ap the depth
using prev.
values depends of
FD scheme in time

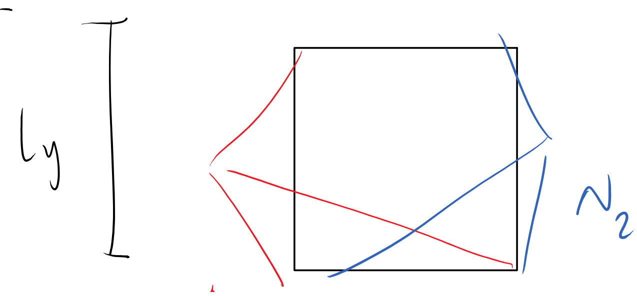
Structure of M for explicit methods for a 2D problem



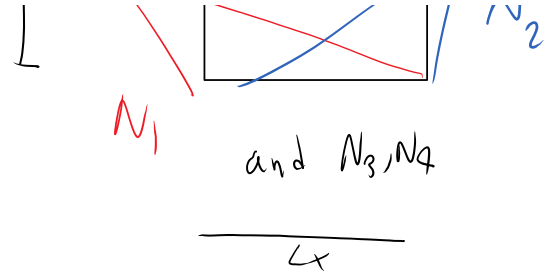


C is 15×15 (15 dofs)

$$C^e = \int_e (N^e)^T C(x) N^e \, dV$$



$$N_1 = \left(1-\frac{x}{L_x}\right)\left(1-\frac{y}{L_y}\right)$$



$$C^e = \int \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} c \,[N_1 \; N_2 \; N_3 \; N_4]\, dv$$

for example $\quad C^e_{11} = \int N_1 N_1 \, dv = C \int_0^{L_x}\int_0^{L_y} \left(1-\frac{x}{L_x}\right)^2 \left(1-\frac{y}{L_y}\right)^2 dx\,dy$

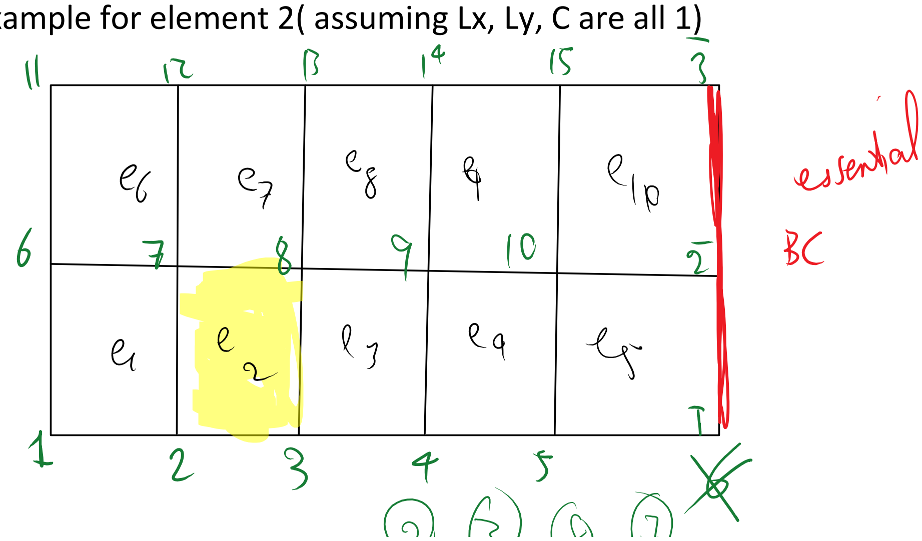
Const capacity

$$\Rightarrow \quad C^e_{11} = \frac{c L_x L_y}{9}$$

$$\boxed{C^e = \frac{c L_x L_y}{36} \begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} \quad \text{CFEM}} \quad (4)$$

we assemble this to global $C_{15\times15}$

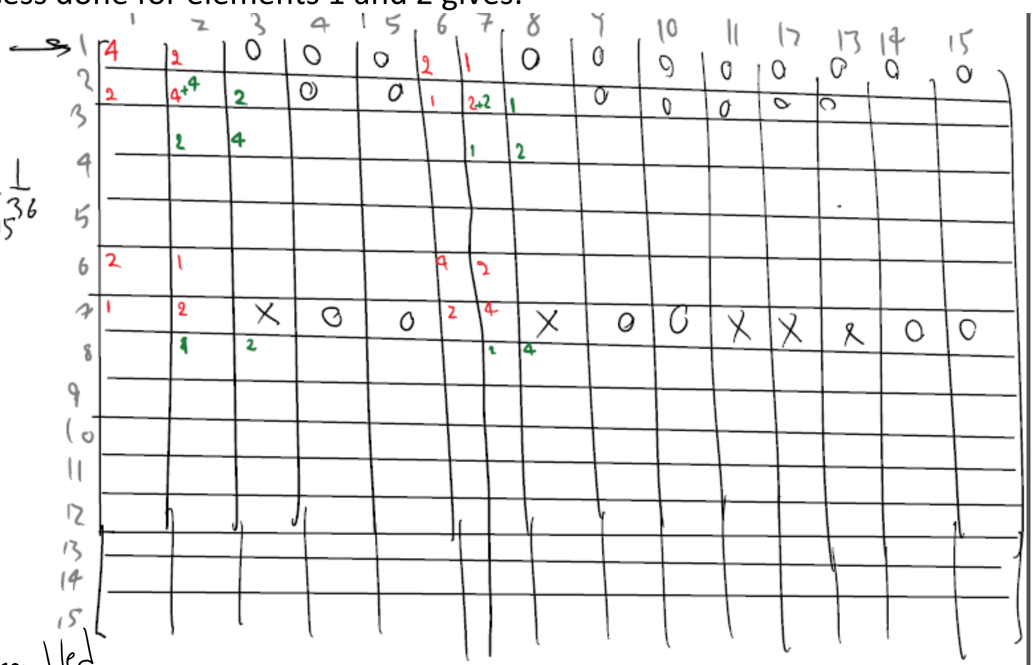
For example for element 2(assuming Lx, Ly, C are all 1)



$$C^{e_2} = \begin{array}{c} \\ 2 \\ 3 \\ 8 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 3 & 8 & 7 \end{array} \\ \frac{1}{36}\begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ & 1 & 2 & 4 \end{pmatrix} \end{array}$$

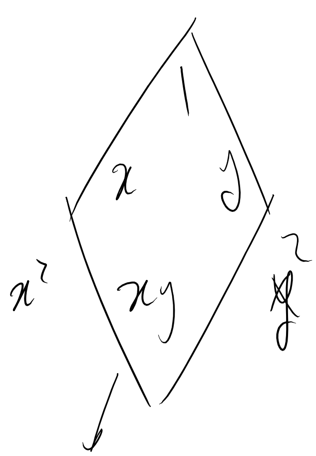
And this is assembled to global C
This process done for elements 1 and 2 gives:

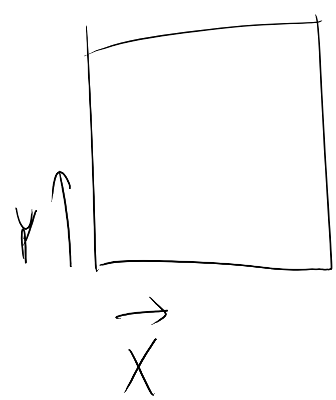
$C_{15\times15} = \frac{1}{36}$



CFEM after e_1, e_2 C^e's are assembled

DG version





2D, p=1 basis is spanned by these 4 monomials ii DG we can use them as basis

same process but

$$N = [1 \quad x \quad y \quad xy]$$

basis functions
rather than shape functions (No δ property)

$$C^e = \int_e N^T c N \, dv \Rightarrow$$

$$C^e = c L_x L_y \begin{bmatrix} 1 & L_x/2 & L_y/2 & L_x L_y/2 \\ & L_x^2/3 & L_x L_y/2 & L_x^2 L_y/6 \\ & & L_y^2/3 & L_x L_y^2/6 \\ & & & L_x^2 L_y^2/9 \end{bmatrix}$$

(6)

DG element matrix $C(M)$

Numbering the global system for DG

e_6 21-24	e_7 25-28	e_8 29-32	e_9 33-36	e_{10} 39-40
e_1 1,2,3,4	e_2 5-8	e_3 9-12	e_4 13-16	e_5 17-20

rest of ∂D is natural BC

essential BC

ONLY :) 40 dofs compared to CFEM!

$$C_e^1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array}$$

$C_e^1 = \frac{2}{3} \; 4$

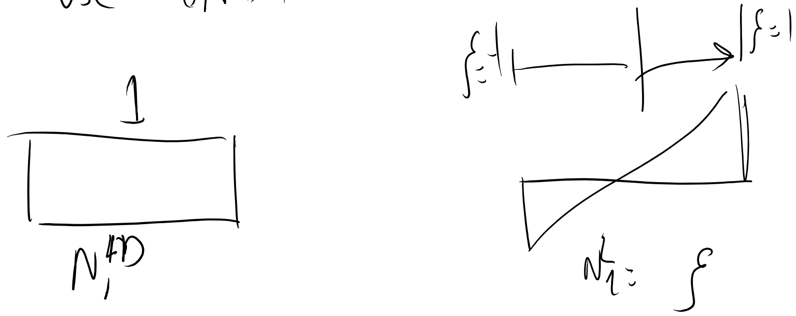
$$C = \begin{array}{c|cccc} & 1\text{-}4 & 5\text{-}8 & \cdots & 37\text{-}40 \\ \hline 1\text{-}4 & C^{e_1} & 0 & 0 & 0 \\ 5\text{-}8 & 0 & C^{e_2} & 0 & 0 \\ & 0 & & \ddots & \\ 37\text{-}40 & 0 & 0 & 0 & C^{e_{10}} \end{array}$$

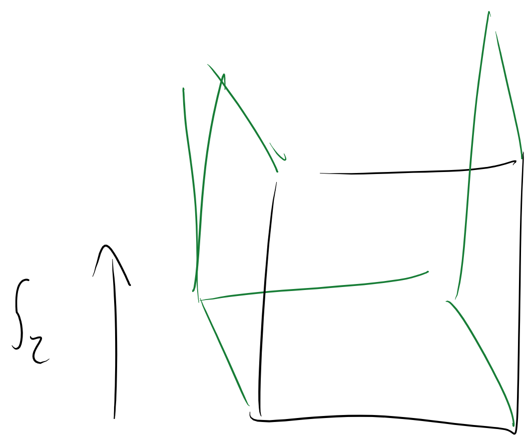
So clearly C is block diagonal

- Is there a way to make this even diagonal?

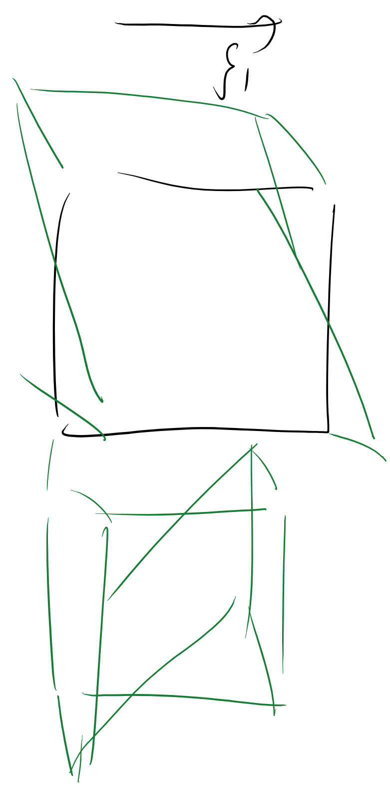
$$C^e_{ij} = \int_e N_i \, \overbrace{c(x)}^{>0} \, N_j \, dx \qquad \langle f,g \rangle = \int_e f \, c \, g \, dv$$

if we want diagonal C we need to use orthonormal basis functions with kernel C





$$N_1 = N_1^{1D}(\xi_1)N_1^{1D}(\xi_2)$$



$$N_2 = N_1^{1D}(\xi_1)N_2^{1D}(\xi_2)$$

N_3

$$N_4 = N_2^{1D}(\xi_1)N_2^{1D}(\xi_2)$$

that will give diagonal C^e (C) global

even better.

Going back to eqn 3:

$$\boxed{(M + \Delta t K)\, a_{n+1} = \Delta t\, f_{ext} + M a_n} \quad (3e)$$

Going back to eqn 3:

$$\underbrace{\left(M + \Delta t K\right)}_{\mathcal{M}_{n+1}} a_{n+1} = \underbrace{\Delta t f_{n+1} + M a_n}_{\mathcal{F}_{n+1}} \qquad (3a) \quad \text{Implicit (BW Euler)}$$

$$\mathcal{M}_{n+1} a_{n+1} = \mathcal{F}_{n+1}$$

$$\underbrace{M}_{\mathcal{M}_{n+1}} a_{n+1} = \underbrace{\left(M - \Delta t K\right) a_n + \Delta t f_n}_{\mathcal{F}_{n+1}} \qquad (3b) \quad \text{Explicit FW Euler}$$

$$\mathcal{M}_{n+1} a_{n+1} = \mathcal{F}_{n+1}$$

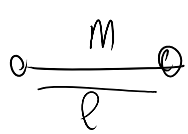
If an explicit method is used, only M (C = M in this case) appears on the LHS so its form determines solution complexity

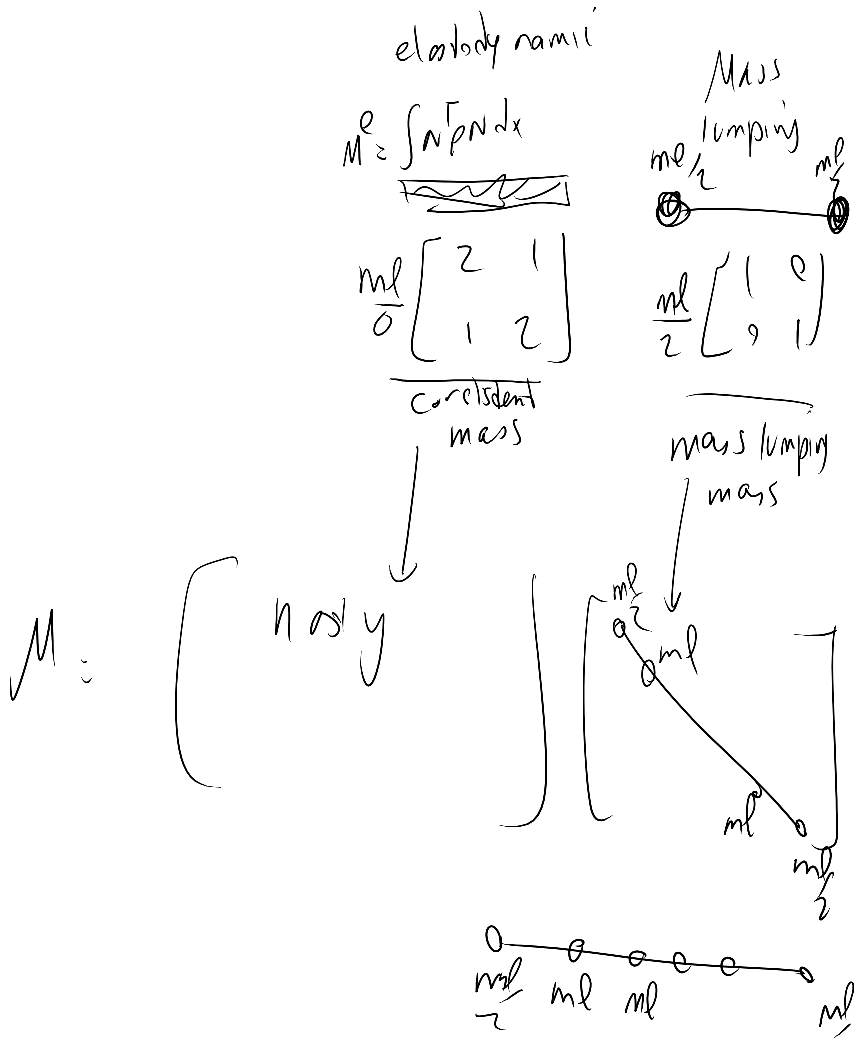
- DG (block diagonal -> can make it diagonal) M : much faster than CFEM non-block diagonal M for solution. (This is even considering the fact that DG has more dofs)

For implicit approach K appears on the LHS. Even though K looks nicer for DG (because of edge rather nodal coupling) still we have to solve a nontrivial matrix equation.

=> That is why the majority of DG methods for hyperbolic and parabolic PDEs use explicit solvers so they can take advantage of their block-diagonal LHS matrix.

It can be argued that for CFEMs we do mass lumping so even in that case M is diagonal for an explicit solver.





So yes, the argument holds that in CFEM mass lumping is very common and in fact it works pretty well.

Especially, we note that explicit integrators tend to shorten the frequency, and mass lumping has the opposite effect. In fact, this is a "match made in heaven":) because not only mass lumping results in diagonal M in CFEM but also do to this counter-acting frequency error effects, results in decent dispersion errors.

However, when the spatial order of elements increases, in order to preserve order of accuracy for wave propagation problems (as opposed to structural dynamics problem), the low order of time integrators such as Euler method affects overall order of accuracy of the method. Solution to that is obviously to increase the order of accuracy of the time integrator. However, even with this the "approximation" and "deviation from Weak Form Consistent Mass matrix" still affects the order of accuracy when mass lumping is used. So, if element order is high, then mass lumping doesn't seem to be a good idea and DG methods offer a much better option (remember that they are in general better for high order solutions anyway in terms of average dof/element).