The other term:

\[ \left[ \begin{array}{c}
\mathcal{C} \\
\mathcal{F} \\
\mathcal{G} \\
\mathcal{H}
\end{array} \right] = \alpha \left[ \begin{array}{cccc}
1 & 1 & -1 & 0 \\
0 & 1 & 1 & -1
\end{array} \right] \]

4x4 stiffness term

\[ \begin{align*}
\mathbf{q}^- &= \begin{bmatrix} 0 & -\frac{k}{h} & 0 & 0 \end{bmatrix} \\
\mathbf{q}^+ &= \begin{bmatrix} 0 & 0 & 0 & -\frac{k^T}{h^T} \end{bmatrix} \\
\mathbf{q}^0 &= \frac{1}{2} (\mathbf{q}^- + \mathbf{q}^+ - \mathbf{q}^+) \\
\end{align*} \]

Stiffness term:

\[ \left[ \begin{array}{ccc}
\mathcal{C} \\
\mathcal{R} \\
\mathcal{R} \\
\mathcal{R}
\end{array} \right] = \left( \begin{array}{c}
1 \\
-1 \\
0 \\
0
\end{array} \right) \begin{bmatrix} 0 & -\frac{k}{h} & 0 & -\frac{k^T}{h^T} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix} \]

Finally:

\[ -\mathbf{e}^T \mathbf{q}^0 \mathbf{1} = (-\mathbf{e}) \sqrt{\frac{k}{h^2}} \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix} \]
\[ K_{\text{interface}} = k_1 + k_2 + k_3 \]

After assembly of interior element:

\[ K \begin{bmatrix} F \end{bmatrix} = M \]

\[ K = 1 \text{ if } Q \neq 0 \]
<table>
<thead>
<tr>
<th>interior element</th>
<th>( \sqrt{ } )</th>
<th>( \sqrt{ } ) only if ( Q \neq 0 )</th>
<th>( \sqrt{ } )</th>
</tr>
</thead>
<tbody>
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<td>( \sqrt{ } )</td>
<td>( \sqrt{ } )</td>
<td></td>
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<tr>
<td>natural BC</td>
<td>( X )</td>
<td>( \sqrt{ } )</td>
<td></td>
</tr>
<tr>
<td>initial</td>
<td>( \sqrt{ } )</td>
<td>( X )</td>
<td></td>
</tr>
</tbody>
</table>

For elliptic problem we get

\[
K \alpha = F \\
\text{n elements } \rightarrow \text{2n defs for linear elements}
\]

for parabolic Case

\[
C \dot{\alpha} + K \alpha = F
\]

Comment on time marching

IC function \( \alpha(t) \) \( \text{or } \alpha(0) \) \( \alpha @ \text{time } 0 \)

\[
C \left( \frac{\alpha_{n+1} - \alpha_n}{\Delta t} \right) + K \alpha_n = F_n
\]

\( n = 1 \) \( \alpha_n \) is obtained from IC

\( T = 1 \) \( Q = 1 + \Delta t \)
For parabolic PDE, K is the same as elliptic one:

New additions are:

1. Calculate M (pretty easy)
2. Time marching:
   a. Compute $F_n$ for each time
   b. Solve

$$C_{n+1} = C_n - \Delta t \, \text{K} \, C_n + F_n$$

$$a_{n+1} = a_n + \left( MC^{-1} K \right) a_n + \left( C^{-1} \right) F_n$$

$$C = \begin{bmatrix} C_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_{N1} \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} C_{11}^{-1} \\ \vdots \\ C_{N1}^{-1} \end{bmatrix}$$

$\text{Diag}$ holds constant $MC^{-1} K$

else $C^{-1} K C^n K C^n K (\Delta t \text{tan})$
Physical fluxes for parabolic PDEs ...

See
Lorcher_2008_An explicit discontinuous Galerkin scheme
with local time-stepping for general unsteady diffusion
equations.pdf

---

Another point on numerical fluxes for parabolic / elliptic PDEs

\[
\theta_t \hat{q} = \int_T \hat{q} \psi d\Omega_i
\]

\[
q^\alpha = \int_T \hat{q} \psi d\Omega_i + \sum \Delta \Omega_i \alpha
\]

The contribution from alpha term is similar to
The contribution from alpha term is similar to J0 term in interior penalty methods.

\[ J_0(v, w) = \sum_{n=0}^{N} \alpha \frac{\sigma_0}{h_{n-1, n}} [v(x_n)][w(x_n)] \]

J0 term in interior penalty methods

for dimensional consistency, \( \alpha = \frac{C}{h_{\text{element}}} \)

---

Notion of coercivity for elliptic operators

The weak statement was:

\[ B(v, \Gamma) = L(\Gamma) \Rightarrow \]

\[ \sum_{e} \int_{e} \nabla \hat{u} \cdot \nabla T dv + \sum_{e} \int_{e} (\hat{T} q - e T \hat{q}) \text{ ands} \]

\[ + \sum_{\Gamma e+} \int_{\Gamma e+} (\hat{T} n q - e [T] \hat{q}) - \sum_{\Gamma e-} \int_{\Gamma e-} (\hat{T} n q - e [T] \hat{q}) + \alpha \sum_{\Gamma e} \| [T] \|_{\Gamma e} \]

\[ = L_e(\Gamma) + \sum_{e} L_e(\Gamma) + \sum_{\Gamma e} L_e(\Gamma) \]
For constant $T$ per element $B(T, T) = 0$ if $e \in$ and $e \in$ this case may not be feasible depending on the boundary condition.

$$B(T, T) =$$

$$\sum_{e} \int \nabla T \cdot \nabla T d\nu + \sum_{e} \int (T_{0} - \epsilon T_{0}) d\nu$$

$$+ \sum_{\Gamma} \int \left( T_{\Gamma} T_{\Gamma} - \epsilon \left[ T_{\Gamma} \right] \right) + \alpha \left[ \left[ T_{\Gamma} \right] \right]$$

$$E = 1, B(T, T) = B_{E}(T, T) = \sum_{e} \int \nabla T \cdot \nabla T + \frac{\epsilon}{\Gamma_{e}} \int \left[ T_{\Gamma} \right] \geq 0$$

For constant $T$ per element $B(T, T) = 0$ if $\alpha = 0$

For constant for the whole domain $B(T, T) = 0$ - this case may not be feasible depending on the boundary condition.

$$B(T, T) \geq 0 \quad \forall T$$

$E = -1, 0$

$0 / \setminus T$ and $\setminus 0$
In this case $B(T, T)$ is not necessarily $\geq 0$

Coercivity of the bilinear form is defined as

$$\exists \lambda > 0 \quad \forall T \quad B(T, T) \geq \lambda |T|^2$$

Uses of coercivity:

1. Uniqueness

   $T_1, T_2$ are solutions

   $\forall T \in \text{domain, } B(f, T_1) = L(f)$

   $\forall T \in \text{domain, } B(f, T_2) = L(f)$

   Subtracting $B$ is bilinear

   $\forall T \in \text{domain, } B(f, T_1 - T_2) = 0$
As can be seen, coercivity can be used to prove uniqueness. The use of it is beyond this simple exercise and for elliptic / parabolic problems is often an integral part of stability & convergence proofs.

Relation of coercivity and form of stiffness matrix

\[ B(\mathbf{T}, \mathbf{T}) = L(\mathbf{T}) \rightarrow K\mathbf{a} = \mathbf{f} \]
\[ B(T,T) \approx L(T) \Rightarrow K_a = f \]

Positive matrix

All eigenvalues of \( K \) are \( > 0 \)

\[ K = \text{sym} K + \text{skew} K \]

\[ \frac{1}{2} (K + K^T) + \frac{1}{2} (K - K^T) \]

\[ a \cdot K a = a \cdot \text{sym} K a + a \cdot (\text{skew} K) a \]

\[ a \cdot K a \geq 0 \Rightarrow a \cdot (\text{sym} K) a \geq 0 \]

Positive definite

\[ \geq 0 \text{ positive} \]

Coercivity \( \exists \lambda > 0 \) \( \exists B(T,T) \geq \lambda |T| \)
Coercivity: 
\[ J > 0 \Rightarrow \exists \eta \in \mathbb{R}^+, \|u\| \leq \eta \|u\| \]

\[ \Rightarrow K \text{ has all positive eigenvalues} \]

in that case \( \lambda = \min \left( \lambda_i \right) \) eigenvalues of \( K \) works.

Do we really coercivity for uniqueness 
\( (\lambda_i > 0) \)

\( Ka = F \)

No, we only need \( K \) be invertible \( (\det K \neq 0) \)

\( \lambda_i \neq 0 \)

So, \( K \) can be indefinite and we still have unique solution, but the scheme still may not be convergent.

- If \( \epsilon = -1 \) and \( \sigma^0 = \sigma^1 = 0 \), the resulting method is called the global element method, introduced in 1979 by Delves and Hall [43]. However, the matrix associated with the bilinear form is indefinite, as the real parts of the eigenvalues are not all positive and thus the method is not stable.