2018/02/19

Monday, February 19, 2018 11:37 AM

Last part:

Interior interfaces:

x^- $\quad h^-, k^-$ $\qquad\qquad$ x^+ $\quad h^+, k^+$

e^- $\qquad\qquad\qquad\qquad$ e^+

$T^- = [1 \;\; x^-]$, $\quad x^- = \frac{X - X_0^-}{h^-}$ $\qquad$ $T^+ = [1 \;\; x^+]$

$T'^- = [0 \;\; \frac{1}{h^-}]$ $\qquad$ $T'^+ = [0 \;\; \frac{1}{h^+}]$

$q^- = -k^- T'^- = [0 \;\; -\frac{k^-}{h^-}]$ $\qquad$ $q^+ = [0 \;\; -\frac{k^+}{h^+}]$

dofs 1&2 $\quad x^- = 1$ $\qquad$ $x^+ = 0$ $\quad$ dofs 3 4

$T^- = [1 \;\; 1]$ $\qquad$ $T^+ = [1 \;\; 0]$

$K_{4\times 4}$ stiffness add beformed $\quad q^- = [0 \;\; -\frac{k^-}{h^-}]$ $\qquad$ $q^+ = [0 \;\; -\frac{k^+}{h^+}]$

$$+ \sum_{\Gamma_{e\bar{e}^t}} \int [[\hat{T}]] \left(\{\{q\}\} + \bar{\alpha} [[T]] \right) ds + \varepsilon \int \{\{\hat{q}\}\} [[T]] ds$$

$$[[\hat{T}]] = T^- - T^+ \quad \begin{array}{l} \text{dofs } e^- \\ \\ \text{dofs of } e^+ \end{array} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$[[\hat{T}]]\,\alpha\,[[T]] = \underbrace{\alpha \begin{pmatrix}1\\1\\-1\\0\end{pmatrix}\begin{bmatrix}1 & 1 & -1 & 0\end{bmatrix}}_{4\times 4 \text{ stiffness term}} \quad K_1$$

The other term:

$$[[\hat{T}]]\,\{\{q\}\}$$

$$q^- = \begin{bmatrix}0 & -\frac{k^-}{h^-} & \Big| & \overbrace{0 \quad 0}^{\text{for element } +}\end{bmatrix}$$

$$q^+ = \begin{bmatrix}\underbrace{0 \quad 0}_{\text{for element } e^-} & \Big| & 0 & -\frac{k^+}{h^+}\end{bmatrix}$$

$$\{\{q\}\} = \frac{1}{2}(q^- + q^+)$$

$$\{\{q\}\} = \frac{1}{2}\begin{bmatrix}0 & -\frac{k^-}{h^-} & \Big| & 0 & -\frac{k^+}{h^+}\end{bmatrix}$$

stiffness term : $[[\hat{T}]]\{\{q\}\} = \begin{pmatrix}1\\1\\-1\\0\end{pmatrix}\left(\frac{1}{2}\right)\begin{bmatrix}0 & -\frac{k^-}{h^-} & \Big| & 0 & -\frac{k^+}{h^+}\end{bmatrix} \quad K_2$

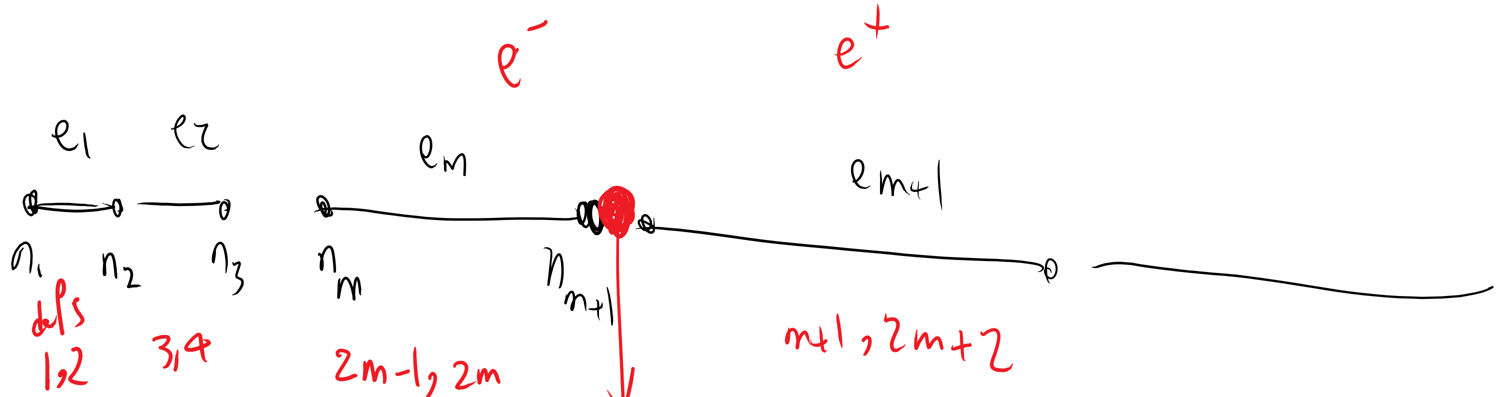
Finally

$$-\varepsilon\{\{q\}\}\,[[T]] = (-\varepsilon)\left(\frac{1}{2}\right)\begin{pmatrix}0\\-\frac{k^-}{h^-}\\0\\\ldots\end{pmatrix}\begin{bmatrix}1 & 1 & -1 & 0\end{bmatrix} \quad K_3$$

$$-\varepsilon\{\{q\}\}[[T]] = (-\varepsilon)\left(\frac{1}{2}\right)\begin{bmatrix} -\frac{k^-}{h^-} \\ 0 \\ -\frac{k^+}{h^+} \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix} \quad K_3$$

$\varepsilon = -1 \rightarrow$ these last two terms are transpose of each other

$$K_{Interface} = K_1 + K_2 + K_3$$



n_{m+1} stiffness

4x4

$$\begin{array}{c|cccc} & 2m-1 & 2m & 2m+1 & 2m+2 \\ \hline 2m-1 & & & & \\ 2m & & & & \\ 2m+1 & & & & \\ 2m+2 & & & & \end{array} \quad K\ interface$$

this is assembled to global K matrix

After assembly of

	K	F	M (parabolic)
interior element	✓	✓ only if $Q \neq 0$	✓

interior element	✓	✓ only if $Q \neq 0$	✓
essential BC	✓	✓	
natural BC	X	✓	
interfaces	✓	X	

For elliptic problem we get

$Ka = F$ n elements $\longrightarrow$ $2n$ dofs for linear elements

for parabolic case

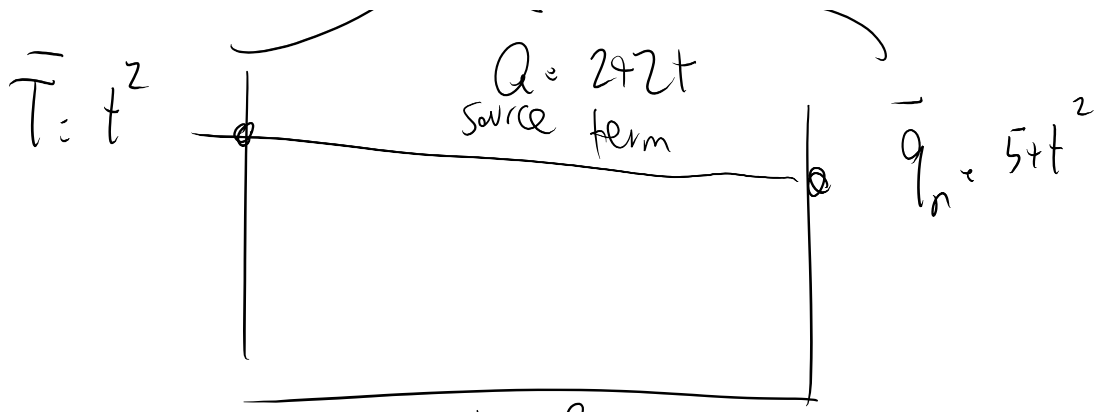
$C\mathring{a} + Ka = F$

Comment on time marching

IC function $a(1)$ or $a(e)$ a @ time 0

$C\left(\frac{a_{n+1} - a_n}{\Delta t}\right) + K a_n = \underbrace{F_n}$ $n = 1$ a_n is obtained from IC

T ^{2} 1 $Q = 242t$



basically F_n needs to be calculated for each time step

For parabolic PDE, K is the same as elliptic one:
New additions are:

1. Calculate M (pretty easy)
2. Time marching:
   a. Compute Fn for each time
   b. Solve

$$C a_{n+1} = C a_n - \Delta t K a_n + F_n$$

$$a_{n+1} = a_n + \underbrace{(\Delta t C^{-1} K)} a_n + (C^{-1}) F_n$$

$$C = \begin{bmatrix} C_{11} & & \\ & C_{22} & \\ & & C_{33} \end{bmatrix} \qquad C^{-1} = \begin{bmatrix} C_{11}^{-1} & & \\ & C_{22}^{-1} & \\ & & C_{33}^{-1} \end{bmatrix}$$

if Δt constant $\Delta t C^{-1} K$

else $C^{-1} K$ CmK CmK (Δtan)

for parabolic PDE stable time step =

$$\underbrace{C_p(\ldots)}_{\text{depends on space order time integration scheme, DG/CFEM..}} \min \left(\frac{h_i^2}{2 \nu_i}\right) \qquad \nu_i = \frac{k_i}{C_i} \qquad \frac{L^2}{T}$$

over the elements

Physical fluxes for parabolic PDEs …
See
Lorcher_2008_An explicit discontinuous Galerkin scheme with local time-stepping for general unsteady diffusion equations.pdf

Appendix A. Derivation of the numerical fluxes

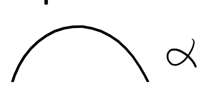
For the computation of the unsteady solution of the initial value problem (2.22) we use Laplace-transformation as described for example in [4]. We solve Eq. (2.22) separately for $\xi_1 < 0$ and $\xi_1 > 0$ and impose compatibility conditions at $\xi_1 = 0$. We denote the Laplace-transformation of $v(\xi_1, t)$ by $w(\xi_1, s)$:

Another point on numerical fluxes for parabolic / elliptic PDEs

$$T^* = \{T\}$$

$$q^* = \{q\} + \tilde{\alpha} k [[T]]$$

The contribution from alpha term is similar to



The contribution from alpha term is similar to

$$J_0(v,w)=\sum_{n=0}^{N}\frac{\sigma^0}{h_{n-1,n}}[v(x_n)][w(x_n)],$$

α

J0 term in interior penalty methods

for dimensional consistency $\alpha = \frac{\sigma}{h}$

element size

Notion of coercivity for elliptic operators

The weak statement was:

$B(\hat{T}, T) = L(\hat{T}) \rightarrow$

$$\sum_e \int_e \nabla\hat{T}\,\kappa\,\nabla T\,dv + \sum_e \int_{\partial e_u} \left(\hat{T}q - \varepsilon T\hat{q}\right)\cdot n\,ds$$

$$+\sum_{\Gamma_{ee^+}} \int_{\Gamma_{ee^+}} \left([[\hat{T}]]\{\{q\}\} - \varepsilon [[T]]\{\{\hat{q}\}\}\right) + \bar{\alpha}[[\hat{T}]][[T]] =$$

$$\sum_e L_i^e(f) + \sum_{\partial e_u} L_u^e(\hat{T}) + \sum_{\partial e_f} L_f^e(\hat{\lambda})$$

$$\sum_e L_i^e(\hat{f}) + \sum_{\partial e_u} L_u^e(\hat{T}) + \sum_{\partial e_f} L_f^e(\hat{\lambda})$$

$$\hat{T} = T$$

$$\boxed{\begin{aligned} B(T,T) = & \sum_e \int_e \nabla T \alpha \nabla T dv + \sum_e \int_{\partial e_u} (T q - \varepsilon T q) \cdot n ds \\ & + \sum_{\Gamma_{ee^+}} \int_{\Gamma_{ee^+}} \left([[T]] \{\{q\}\} - \varepsilon [[T]] \{\{q\}\} \right) + \tilde{\alpha} [[T]] [[T]] \end{aligned}}$$

$\varepsilon = 1$ $\quad B(T,T) = B_{\varepsilon=1}(T,T) = \sum_e \int_e \nabla T \kappa \nabla T + \sum_{\Gamma_{ee^+}} \int \tilde{\alpha} [[T]] [[T]] \geq 0$

For constant T per element B(T, T) = 0 if $\tilde{\alpha} = 0$

For constant for the whole domain B(T, T) = 0 - this case may not be feasible depending on the boundary condition.

$B(T,T) \geq 0 \quad \forall T$

$\varepsilon = -1, 0$

$0 \int \quad \int T q \cdot n ds \quad \int [[T]] [[q]]$

$$B(T,T) = B_{\varepsilon=1}(T,T) + f\left(\sum_{\partial\Omega_N} \int T q \cdot n \, ds + \sum_{\Gamma_{ee'}} \int [[T]] \{\{q\}\}\right)$$

$\Gamma_{ee'}$

$$f = \begin{cases} 1 & \varepsilon = 0 \\ 2 & \varepsilon = -1 \end{cases}$$

In this case B(T, T) is not necessarily >= 0

Coercivity of the bilinear form is defined as

$$\boxed{\exists \lambda > 0 \ni \forall T \qquad B(T,T) \geq \lambda \underbrace{|T|}_{\text{magnitude}}^2}$$

Uses of coercivity:

1. Uniqueness

T_1, T_2 are solutions

in discrete solution space

$\forall \hat{T} \in V \quad B(\hat{T}, T_1) = L(\hat{T}) \qquad T_1$ is a solution

$\forall \hat{T} \in V \quad B(\hat{T}, T_2) = L(\hat{T}) \qquad T_2$ " " "

subtract (B is bilinear)

$\forall \hat{T}$ discrete $\; B(\hat{T}, T_1 - T_2) = 0 \qquad \forall \hat{T}$

$$\forall \hat{T} \in \underset{\text{sln space}}{\text{discrete}} \quad B(\hat{T}, T_1 - T_2) = 0 \qquad \forall \hat{T}$$

$$\Downarrow$$

$$\hat{T} = T_1 - T_2 \qquad B(T_1 - T_2, T_1 - T_2) = 0$$

$$T_1 \in V, \; T_2 \in V \rightarrow T_1 - T_2 \in V$$

$$\underbrace{\lambda}_{\lambda > 0} \underbrace{\| T_1 - T_2 \|}_{\text{magnitude operator}}^2 \leq B(\underbrace{T_1 - T_2}_{T}, \underbrace{T_1 - T_2}_{T}) = 0$$

$$\lambda \| T_1 - T_2 \|^2 \leq 0 \implies \boxed{T_1 = T_2}$$

proof of uniqueness

As can be seen, coercivity can be used to prove uniqueness. The use of it is beyond this simple exercise and for elliptic / parabolic problems is often an integral part of stability & convergence proofs.

Relation of coercivity and form of stiffness matrix

$$B(\hat{T}, T) = L(\hat{T}) \implies Ka = f$$

$$B(\dot{T}, T) = L(T) \implies Ka = f$$

stiffness comes from bilinear form

$$B(T,T) \geq 0 \iff a \cdot Ka \geq 0$$

positive matrix

all eigenvalues of K are ≥ 0

$$K = \text{sym}K + \text{skew}K$$
$$= \tfrac{1}{2}(K + K^T) \quad \tfrac{1}{2}(K - K^T)$$

$$a \cdot Ka = a \cdot (\text{sym}K)a + \underbrace{\cancel{a \cdot (\text{skew}K)a}}_{0}$$

$$a \cdot Ka \geq 0 \iff a \cdot (\text{sym}K)a \geq 0$$

> 0 positive definite

≥ 0 positive

$$\boxed{\text{Coercivity } \exists \lambda > 0 \ni B(T,T) \geq \lambda |T|^2}$$

Coercivity $\exists \lambda > 0 \ni B(v, v) \leq \lambda \| v \|$

$\equiv$ K has all positive eigenvalues

in that case $\lambda = \min(\lambda_i)$ eigenvalues of K works

Do we really coercivity for uniqueness ($\lambda_i > 0$)

$$Ka = F$$

No, we only need K be invertible ($\det K \neq 0$)

$$\lambda_i \neq 0$$

So, K can be indefinite and we still have unique solution, but the scheme still may not be convergent

- If $\epsilon = -1$ and $\sigma^0 = \sigma^1 = 0$, the resulting method is called the global element method, introduced in 1979 by Delves and Hall [43]. However, the matrix associated with the bilinear form is indefinite, as the real parts of the eigenvalues are not all positive and thus the method is not stable.