2018/02/21

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Hyperbolic PDEs: Discontinuous Galerkin formulation

$$\boxed{C\ddot{U} + d\dot{U} - \underbrace{\nabla\cdot(K\nabla u)}_{\text{or an elliptic operator}} = S}$$ ①

example Solid Mechanics $C \to \rho$

$d \to$ damping

$\nabla\cdot(K\nabla u) \to \nabla\cdot\sigma \qquad \sigma = \underset{\text{stiffness}}{\mathbb{C}}\underbrace{\tilde{\nabla}u}_{\text{strain}}$

We will write (1) in the form of a system of conservation laws:

$$V = \dot{U}$$
$$q = \nabla u$$

first order temporal & spatial derivatives are defined as auxiliary fields

$$\dot{U} + f_{x,x}(U) + f_{y,y}(U) + f_{z,z}(U) = S$$

$$\boxed{\dot{U} + \nabla\cdot F(U) = 0}$$ ②

$$\boxed{\underbrace{U}_{\text{temporal flux density}} + \nabla \cdot F(U) = 0}$$ ②

↘ spatial flux density

For linear systems of conservation law F is a linear function of U

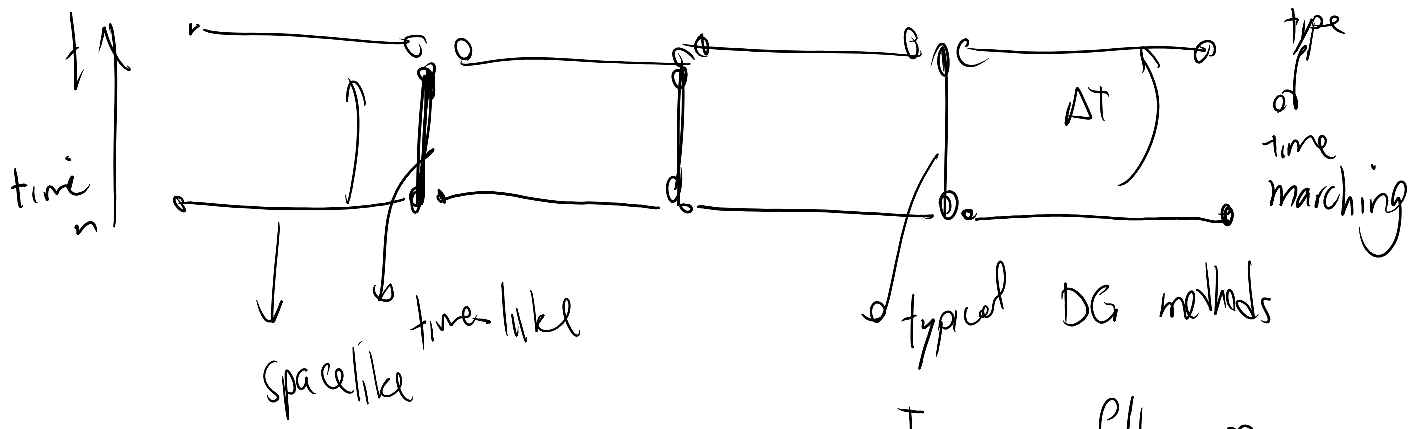
$F_x(U) = A_x U \ \ldots$

$$\boxed{U_{,t} + A_x U_{,x} + A_y U_{,y} + A_z U_{,z} = S}$$ ③ for linear system

so eqn ① can be written as

$$\boxed{\begin{cases} \dot{u} = \overbrace{v}^{\text{source terms}} & (4i) \\ C\dot{v} - \nabla k q = -dv + S & (4ii) \quad \text{balance law ①} \\ \underbrace{\dot{q}}_{\text{temporal}} \quad \underbrace{-\nabla v}_{1} = 0 & (4iii) \end{cases}}$$ ④

derivatives ↓ spatial derivatives



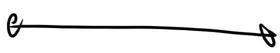
Jump manifolds are only spatial (time-like)

⇓

only the jump in spatial flux appears in the weighted residual

Weighted residual form is:

$$\overset{\text{PDE (diffuse) part}}{\int_e \hat{U}\,(\dot{u} - V)\,dv} + \overset{\text{jump part}}{0}$$



e $\underline{4i}$

$$+\int \hat{V}\left(C\dot{v} - \underbrace{\overset{kq=\delta}{\widetilde{\nabla\cdot kq}}}_{\text{spatial}} - S + dV\right) dv + \int \hat{V}\left[-\overset{\delta^*}{\widetilde{(kq)^*}} - \overset{\delta}{(kq)}\right] ds$$

spatial flux

4ii

$$+\int_e \hat{q}\,(\dot{q}-\underset{\sim}{\nabla v})\,dv + \int_{\partial e}\hat{q}\cdot n(-1)(v^{*}-v)\,ds = 0 \quad 4iii$$

We are interpolating:

u

v

q

3 field formulation (u, v, q) are interpolated

Find $\underline{u, v, q}$ so that for all $\hat{u}, \hat{v}, \hat{q}$

✪

a IF formulation remain even in

$$\cancel{\int_e \hat{u}(\dot{u}-v)\,dv}$$

$$+\int_e \hat{v}(c\dot{v}-dv-\nabla\cdot\sigma-g)\,dv + \int_{\partial e}\hat{v}\cdot(-\sigma^{*}-\sigma)^{n}\,ds$$

$$+\int \cancel{\hat{q}(\dot{q}-\nabla v)}\,dv + \int \hat{q}\cdot n\,(-v^{*}-v)\,ds = 0$$

⑤

3 field formulation

$$\cancel{+\int_e \hat{q}\,(\dot{q}-\underset{\circledast}{\nabla v})\,dv} \quad + \int_{\partial e} \hat{q}\cdot n\,(-v^{*}\underset{\circledast}{-}v)\,ds = 0$$

field formulation

$$\sigma = k\nabla u = kq$$

→ removed in a 1F formulation

$\int \hat{u}(\dot{u}-v)$ can easily be removed

→ It is not needed (acoustic eqn displacement is irrelevant) OP

– Can be done as a post processing part after an element solution is obtained

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Path to a single field formulation (only u is interpolated)

For a single-field formulation we strongly satisfy:

$$\begin{cases} \dot{u} = V \\ \dot{q} = \nabla V \end{cases}$$

In (4) integration by parts (Gauss theorem) removes ⊛ Terms from the boundaries

And modifies ⊛ terms inside.

We'll do this process for a single field formulation.

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by using $\dot{u} = v$ & $\dot{q} = \nabla v$ in ④ we get

$$\int_e \hat{v}\left(C\mathring{v} + dv - f - \nabla\cdot\delta\right) dv + \int_{\partial e} \hat{v}\left(-\delta^* + \delta\right)\cdot\vec{n}\, ds \qquad ⑥$$

$$+ \int_{\partial e} \hat{q}\cdot n\left(-v^* - v\right) ds = 0$$

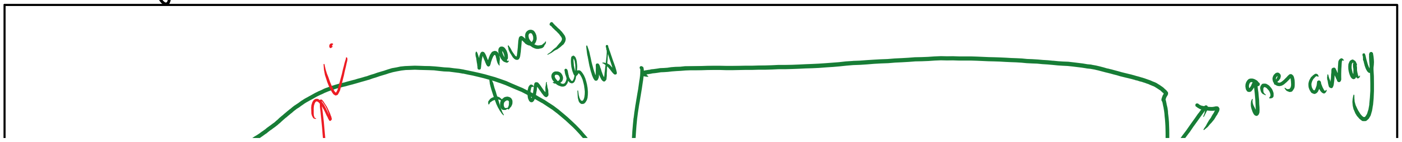
WRS for single field formulation where only u is interpolated.

$\hat{u}$ ✓ $\qquad \hat{u}_i = \phi_i(x) = u_i(x)$

$\hat{v} = \hat{u} = \overset{\circ}{\widehat{u_i}}(x) \qquad u = u_i(x)\, a_i(t)$

$\hat{q} = \nabla\hat{u}$ ✓

Change $\hat{v}$ to $\hat{u}$ in ⑥



moves to weight

goes away

$$\int_e \hat{u}\left(C\,\overset{\dot{v}}{\ddot{u}} + d\,\dot{u} - \cancel{\nabla}\cdot\sigma - S\right)dv + \int_{\partial e} \underbrace{\hat{u}}\left(-\underbrace{\sigma^*}+\sigma\right)\cdot n\,ds$$

$$+\,\overset{\lambda}{\lambda'}\int_{\partial e} \overset{\hat{\sigma}}{\hat{q}}\cdot n\left(-v^*+v\right)ds = 0$$

⑦

$\hat{q} = \nabla\hat{u}$

for dimensional consistency

$$[\lambda' q v] = [u\sigma] \qquad \sigma = \kappa q$$

$$[\lambda'][\cancel{q}][v] = [u][\kappa][\cancel{q}]$$

$$[\lambda']\frac{[u]}{T} = [u][\kappa] \longrightarrow [\lambda'] = T[\kappa]$$

Time

doing the same process with $\hat{\sigma}$ (the better choice)

$$[\lambda q v] = [u q]$$

$$\boxed{[\lambda] = T}$$

λ must have dimension of time

$\boxed{[\lambda] = T}$ λ must have dimension of time

We can create a time scale from element size & wave speed

$$c = \sqrt{\frac{k}{C}} \qquad \lambda = \boxed{\frac{h}{c}} \quad (8)$$

if scalar

wave speed

maximum possible time step

Finally, we do integration by parts (Gauss theorem) to get

$$\int_e \left[\hat{u}\left(C\ddot{u} + d\dot{u} - g\right) + \nabla\hat{u}\,\sigma\right] dv + \int_{\partial e} \hat{u}\left(-\hat{\sigma}\cdot n\right) ds + \lambda \int_{\partial e} \hat{\sigma}\cdot n\left(-\hat{v} + v\right) ds = 0 \quad (9)$$

inhomogeneity, $k\nabla u$ doesn't hurt us

$\lambda = 1$ (non dimensionally consistent)

or $\lambda = \frac{h}{c}$

$$u = \sum u_i(x)\, a_i(t) = U a \qquad U = [u_1(x) \cdots u_n(x)]$$

$$u = \sum u_i(x)\, a_i(t) = U a \qquad U = \lfloor \underbrace{U_1(x) \cdots U_n(x)}_{\text{interpolated functions within the element}} \rfloor$$

$$\dot u = \sum u_i(x)\, \dot a_i(t) = U \dot a$$

$$\ddot u = \sum u_i(x) \cdot \ddot a_i(t) = U \ddot a$$

$$\hat u = \hat U$$

$$\sigma = k \cdot \nabla u \rightarrow \sigma = k \nabla (U a) =$$

Plugging into (9) we obtain:

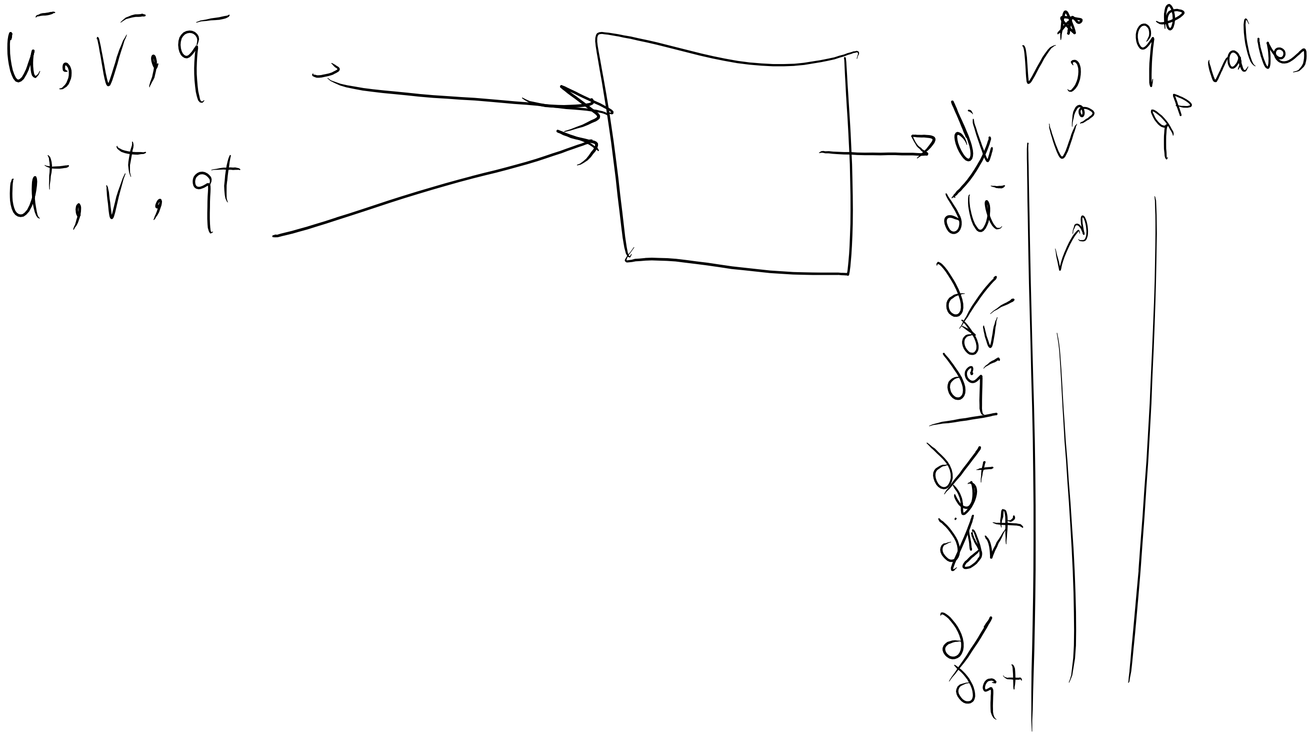
$$\int \hat u C \hat u \, dv \qquad \int \hat u\, d\, \dot u\, dv \qquad \int \nabla \hat u \cdot \sigma\, dv \qquad -\int \hat u f\, dv$$

$$\underbrace{\left(\int_e U^T C U\right)}_{m^e} \ddot a + \underbrace{\left(\int_e U^T d\, U\right)}_{K_b^e} \dot a + \underbrace{\left(\int_e \nabla U^T k \nabla U\right)}_{K_b^e} a - \int_e U^T f\, dv + \int_{\partial e} \hat U (-\sigma^* \cdot n)\, ds + \lambda \int_{\partial e} \hat \sigma \cdot n (-v^* - v)\, ds = 0$$

$$m^e \ddot a + \underbrace{K_b^e}_{\text{damping matrix}} \dot a + K_b^e a - \overset{\overset{\text{source term}}{\uparrow}}{f_r^e} + \int_{\partial e} \hat u (-\sigma^* \cdot n)\, ds + \lambda \int_{\partial e} \hat \sigma \cdot n (-v^* + v)\, ds = 0 \qquad ⑩$$

The best practice is to keep options open for star values.

We write a black-box type function that computes values and derivative of star values in general.



⑪

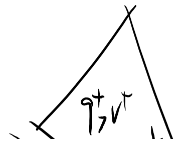
We will solve (10) in 1D

Num. Flux option:

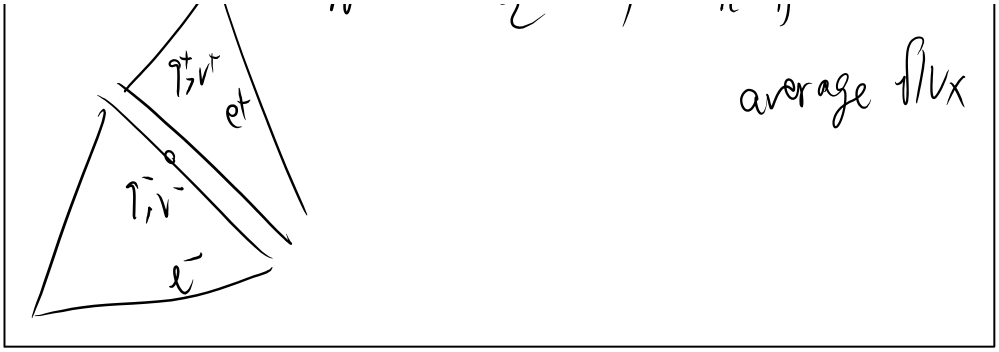
$$q^*_A = \left(\frac{q^- + q^+}{2}\right) = \{\{q\}\}$$

$$v^*_A = \left(\frac{v^- + v^+}{2}\right) = \{\{v\}\}$$

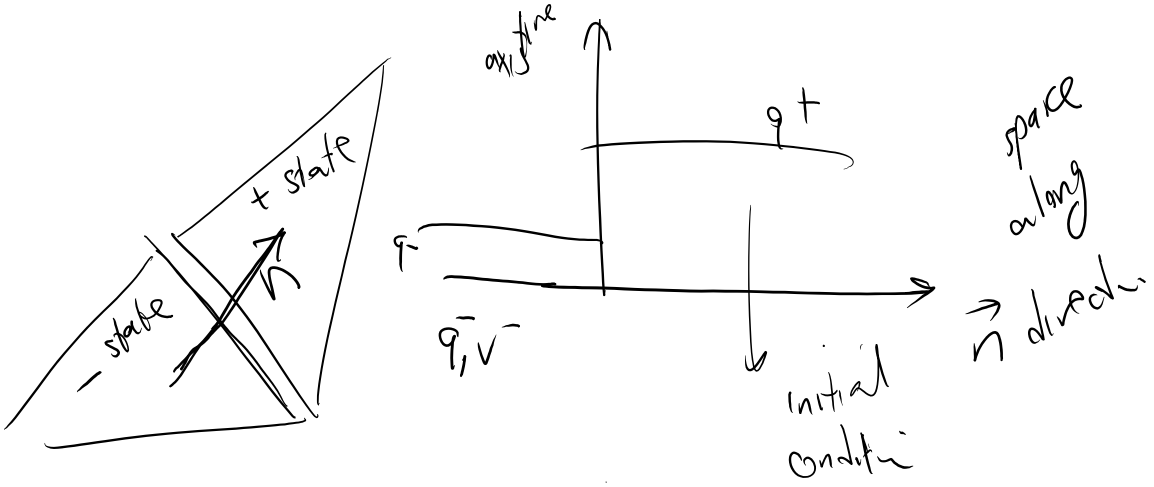
⑫



average flux



We can also solve the so-called Riemann problem for this PDE



We solve this local 1 dimensional PDE to obtain Riemann solutions