2018/02/26

Monday, February 26, 2018 11:39 AM

$$\int \hat{u} C \hat{u} \, dv \qquad \int \hat{u}\, d\, u \, dv \qquad \int \nabla \hat{u} \cdot \sigma \, dv \qquad -\int \hat{u} f \, dv$$

$$\boxed{\underbrace{\left(\int_e U^T C U\right)}_{m^e} \ddot{a} + \underbrace{\left(\int_e U^T d\, U\right)}_{K_b^e} \dot{a} + \underbrace{\left(\int_e \nabla U^T \kappa \nabla U\right)}_{K_b^e} a - \int_e U^T f \, dv + \int_{\partial e} \hat{U}\left(-\sigma^* \cdot n\right) ds + \lambda \int_{\partial e} \hat{\sigma} \cdot n \left(-v^* - v\right) ds = 0} \quad (1)$$

Weighted residual statement from the last time

Deriving matrices for linear elements
Interior contributions:

$$m^e = \int_e U^T C U \underbrace{dv}_{dX \text{ in 1D}}$$

1D, p=1 element

$$= \int \begin{bmatrix} 1 \\ x \end{bmatrix} C \begin{bmatrix} 1 & x \end{bmatrix} (h\, dx)$$

$$x = \frac{X - x_0}{h}$$

$$U = \begin{bmatrix} 1 & x \end{bmatrix}$$

$$\nabla U = \frac{dU}{dX} = \begin{bmatrix} 0 & \frac{1}{h} \end{bmatrix}$$

$$m^e = Ch \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

$\lfloor 2 \quad\quad s \rfloor$

$$C_b^e = \int U^T d \, U \, dX = dh \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

C_b^e ↓ from bulk (interior) of the element

d ↓ damping coefficient

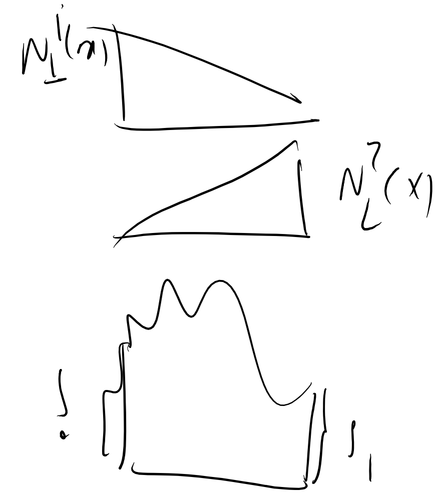
$$K_b^e = \int \nabla U^T \kappa \nabla U \, dX = \int_0^1 \begin{bmatrix} 0 \\ \frac{1}{h} \end{bmatrix} \kappa \begin{bmatrix} 0 & \frac{1}{h} \end{bmatrix} (h \, dx)$$

$$= \frac{\kappa}{h} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Force term from interior:

$$\int_e U^T s \, dV = \Bigg(\underbrace{\int \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} N_L^1(x) & N_L^2(x) \end{bmatrix} h \, dx}_{r_Q} \Bigg) \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \qquad s = s_1 N_L^1(x) + s_2 N_L^2(x)$$

$$= r_Q \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \qquad r_Q = h \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

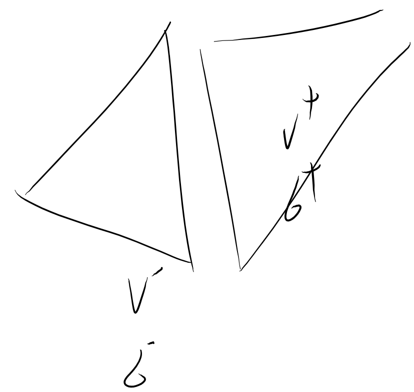


Boundary terms require the computation of star values

$$\int_{\partial e} U\left(-\sigma^{*}\cdot n\right) ds \;+ \lambda \int_{\partial e} \sigma\cdot n\left(-V^{*}-V\right) ds$$

$V^{*} = ?$

$\sigma^{*} = ?$

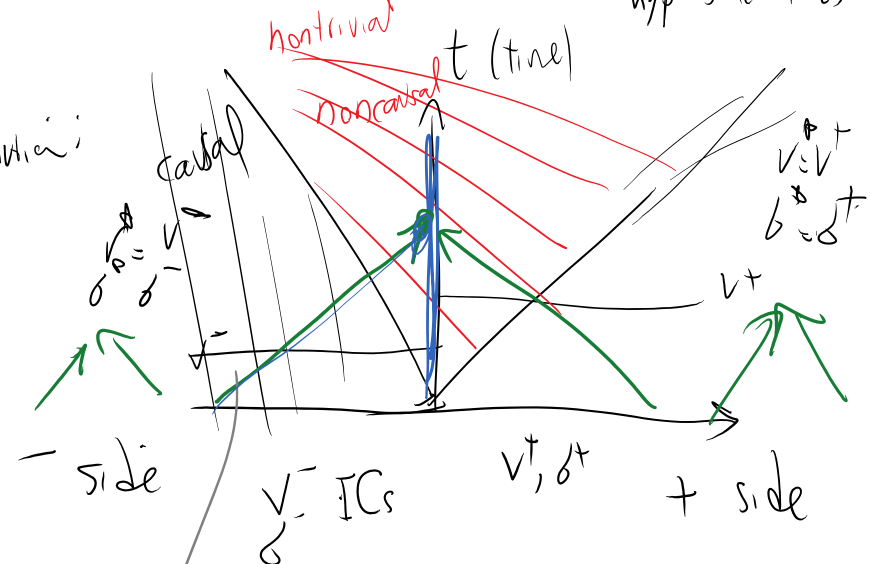


Choices are:

Average flux
$$\begin{cases} V^{*}_{A} = \dfrac{V^{-} + V^{+}}{2} \\ \sigma^{*}_{A} = \dfrac{\sigma^{-} + \sigma^{+}}{2} \end{cases}$$

fine for hyperbolic PDEs

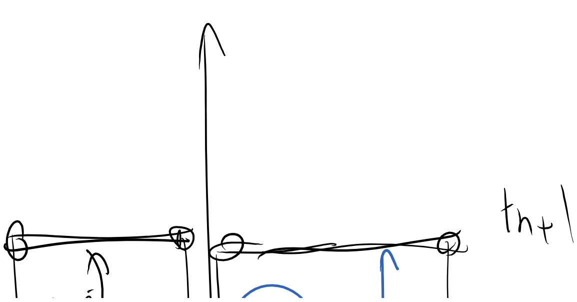
Riemann Solution:

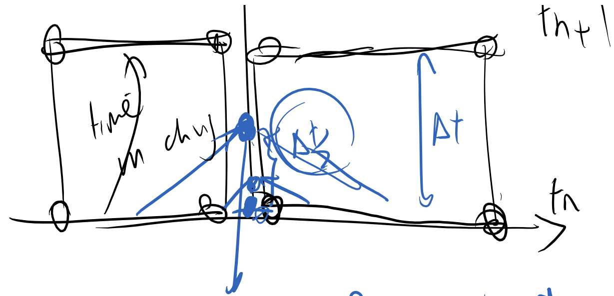


we solve this simple IVP to obtain Riemann solution on vertical line

on characteristic lines, characteristic values are constant (source term = 0)

or we solve a simple ODE





or we solve a simple ODE

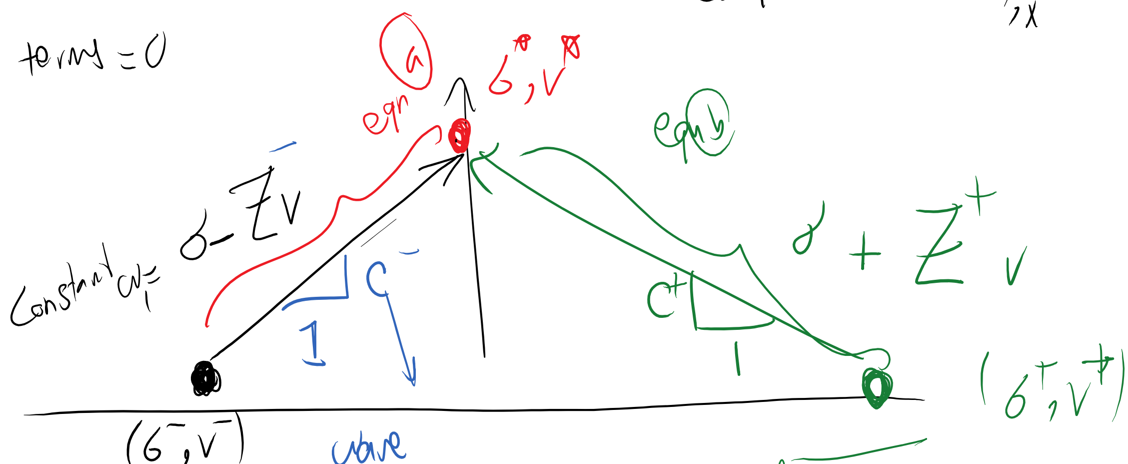
say we compute * values here

$$C\ddot{u} + d\dot{u} - (k u_{,x})_{,x} = f$$

source terms

What are the characteristics for this simple PDE

for source terms $= 0$



Z = impedance $= \sqrt{kC} = c\,C$

c: wave speed; C: coefficient of $\ddot{u}$

wave speed $\sqrt{\dfrac{k^-}{C^-}}$ — side

$c^+ = \sqrt{\dfrac{k^+}{C^+}}$

$Z^+ = c^+ C^+$ +

$$\begin{cases} \sigma^- - Z^- v^- = \sigma^* - Z^- v^* & \text{eqn a} \\ \sigma^+ + Z^+ v^+ = \sigma^* + Z^+ v^* & \text{eqn b} \end{cases}$$

$\underbrace{\qquad\qquad}$

Knowns

Z^-, Z^+ material properties

$\sigma^\mp, v^\mp$ Initial conditions

unknowns

2 eqns, 2 unknowns:

We solve this to obtain:

$$\sigma^* = \left(\frac{Z^- \sigma^+ + Z^+ \sigma^-}{Z^- + Z^+} \right) + \frac{Z^- Z^+}{Z^- + Z^+} \left(v^+ - v^- \right)$$

$$v^* = \frac{1}{Z^- + Z^+} \left(\sigma^+ - \sigma^- \right) + \frac{Z^- v^- + Z^+ v^+}{Z^- + Z^+}$$

added jump terms from the Riemann solution

what if the two materials have the same impedance

$$\sigma^* = \frac{\sigma^- + \sigma^+}{2} + \frac{Z}{2} \left(v^+ - v^- \right)$$

$$v^* = \frac{1}{2} \left(\sigma^+ - \sigma^- \right) + \left(\frac{v^- + v^+}{2} \right)$$

(2)

average flux terms

General expression

$$\sigma^* = \underbrace{\Sigma_{\sigma^-}}_{\text{number}} \sigma^- + \Sigma_{\sigma^+} \sigma^+ + \Sigma_{v^-} v^- + \Sigma_{v^+} v^+$$

$$v^* = V_{\sigma^-} \sigma^- + V_{\sigma^+} \sigma^+ + V_{v^-} v^- + V_{v^+} v^+$$

3

Average flux

$$\begin{cases} \Sigma_{\sigma^-} = \frac{1}{2} & \Sigma_{\sigma^+} = \frac{1}{2} & \Sigma_{v^-} = 0 & \Sigma_{v^+} = 0 \\ V_{\sigma^-} = 0 & V_{\sigma^+} = 0 & V_{v^-} = \frac{1}{2} & V_{v^+} = \frac{1}{2} \end{cases}$$

(R) Riemann fluxes

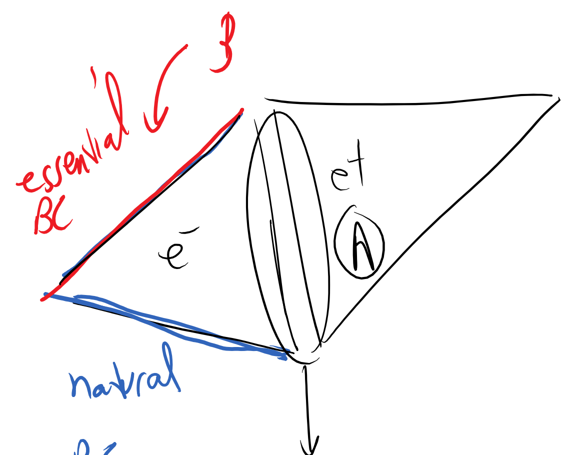
$$\begin{cases} \Sigma_{\sigma^-} = \dfrac{Z^+}{Z^- + Z^+} & \Sigma_{\sigma^+} = \dfrac{Z^-}{Z^- + Z^+} & \Sigma_{v^-} = \dfrac{-Z^- Z^+}{Z^- + Z^+} & \Sigma_{v^+} = -\Sigma_{v^-} \\ V_{\sigma^-} = \dfrac{-1}{Z^- + Z^+} & V_{\sigma^+} = -V_{\sigma^-} & V_{v^-} = \dfrac{Z^-}{Z^- + Z^+} & V_{v^+} = \dfrac{Z^+}{Z^- + Z^+} \end{cases}$$

For nonlinear PDEs (that is in general)

$$\boxed{F^* = f(F^-, F^+)}$$

$$\underbrace{\int_{\partial e} \hat{U}(-\sigma^* \cdot n)\,ds}_{I_u} + \underbrace{\lambda \int_{\partial e} \hat{\sigma} \cdot n\,(-v^* - v)\,ds}_{I_\sigma}$$

Boundary integrals



Case A: Interior of the domain contribution

$$\int_{\Gamma_{e^-e^+}} \hat{U}^- (-\sigma^* \cdot n^-)\,ds + \lambda^- \int_{\Gamma_{e^-e^+}} \hat{\sigma}^- \cdot n^- (-v^* + v^-)\,ds$$

$$+ \int_{\Gamma_{e^-e^+}} \hat{U}^+ (-\sigma^* \cdot n^+)\,ds + \lambda^+ \int_{\Gamma_{e^-e^+}} \hat{\sigma}^+ \cdot n^+ (-v^* + v^+)\,ds$$

(C)

contribution from the left side

$\Gamma_{e^-e^+} \subseteq \Gamma^*$

interior interfaces

$\Gamma_{e^- e^+}$

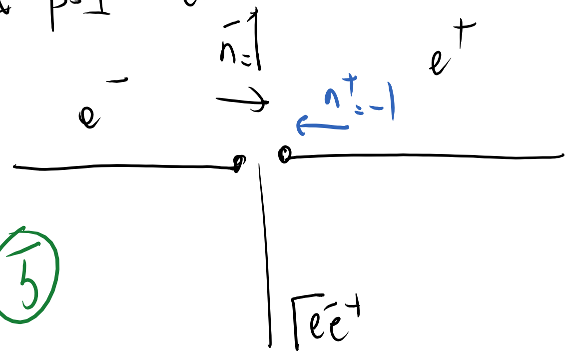
$$B(\hat{U}, V) = \int_{\Gamma_{\delta\delta^+}} \underbrace{(\hat{u}^- n^- + \hat{u}^+ n^+)}_{[[\hat{u}]]} \delta^* \, ds$$

bilinear for interior of the domain

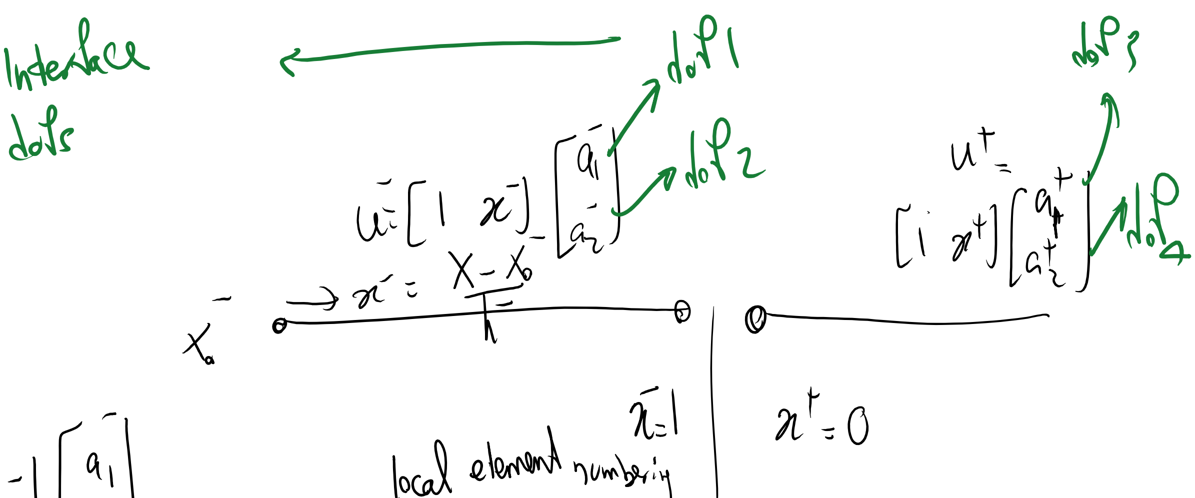
$$+ \int_{\Gamma_{\delta\delta^+}} \left[\lambda^- \hat{\delta}^- \cdot n^- (-v^* + v^-) + \lambda^+ \hat{\delta}^+ \cdot n^+ (-v^* + v^+) \right] ds \quad (4)$$

expression of (4) for 1D & p=1 elements

$$\boxed{ \mathcal{B}_{\Gamma_{e e^+}} = \overbrace{[[\hat{u}]]\, \delta^*}^{\tilde{I}_{\hat{u}}} + \underbrace{\lambda^- \hat{\delta}^- \cdot n^- (-v^* + v^-) + \lambda^+ \hat{\delta}^+ \cdot n^+ (-v^* + v^+)}_{\tilde{I}_{\hat{\delta}}} } \quad (5)$$



Before evaluating these terms, lets compute field values at the interface



$u^- = [1 \;\; x^-] \begin{bmatrix} a_1^- \\ \dots \end{bmatrix}$

$$u^- = [1 \quad x^-]\begin{bmatrix} a_1 \\ a_2^- \end{bmatrix}$$

$$\nabla u^- = u^-_{,x} = \left[0 \quad \frac{1}{h^-}\right]\begin{bmatrix} a_1^- \\ a_2^- \end{bmatrix}$$

$$\sigma^- = \kappa^- \nabla u^- = \left[0 \quad \frac{\kappa^-}{h^-}\right]\begin{bmatrix} a_1^- \\ a_2^- \end{bmatrix}$$

$$V = \dot{u}^- = \frac{d}{dt}\left([1 \quad x^-]\begin{bmatrix} a_1^- \\ a_2^- \end{bmatrix}\right)$$

$$= [1 \quad x^-]\overbrace{\begin{bmatrix} a_1^- \\ a_2^- \end{bmatrix}}^{\circ}$$

local element numbering

$$u^- = [1 \quad 1]a \qquad u^+ = [1 \quad 0]a$$

$$\sigma^- = \frac{\kappa^-}{h^-}[0 \quad 1]a \qquad \sigma^+ = \frac{\kappa^+}{h^+}[0 \quad 1]a$$

$$V^- = [1 \quad 1]\dot{a} \qquad v^+ = [1 \quad 0]\dot{a}$$

What terms are needed for the interface terms:
From equation 5, we need

needed for v^p, σ

$$[\![\hat{u}]\!], V^-, V^+, \sigma^-, \sigma^+$$

$$\lambda^- \hat{\sigma} \cdot n^-, \lambda^+ \hat{\sigma}^+ \cdot n^+$$

σ^p

v^p

$n^- = 1 \qquad n^+ = -1$

$a_{4\times1}$ dofs of the interface

$x^- = 1 \qquad x^+ = 0$

$$u^- = [1 \quad 1 \mid 0 \quad 0]a_e \qquad u^+ = [0 \quad 0 \mid 1 \quad 0]a$$

$$[\![u]\!] = u^- \cdot n^- + u^+ \cdot n^+ = u^- - u^+ = [1 \quad 1 \mid -1 \quad 0]a$$

$$V^- = [1 \quad 1 \mid 0 \quad 0]\dot{a} \qquad V^+ = [0 \quad 0 \mid 1 \quad 0]\dot{a}$$

$$\lambda^- \hat{\sigma} \cdot n^- = [0 \quad \kappa^- \lambda \mid 0 \quad 0 \quad \ldots \qquad \lambda^+ \hat{\sigma}^+ \cdot n^+ [\ldots \quad \lambda^+ \kappa^+$$

$$\lambda^- \hat{\sigma}^- \cdot \underbrace{n^-}_{1} = \left[0 \quad \frac{K^-\lambda}{h^-} \,\middle|\, 0 \quad 0\right] \qquad \lambda^+ \hat{\sigma}^+ \cdot \underbrace{n^+}_{-1} = \left[0 \quad 0 \,\middle|\, 0 \quad \frac{\lambda^+ K^+}{h^+}\right]$$

$$\lambda^- \hat{\sigma}^- \cdot n^- + \lambda^+ \hat{\sigma}^+ \cdot n^+ = \left[0 \quad \frac{K^-\lambda^-}{h^-} \,\middle|\, 0 \quad -\frac{\lambda^+ K^+}{h^+}\right]$$

$$\sigma^- = \left[0 \quad \frac{K^-}{h^-} \,\middle|\, 0 \quad 0\right] a \qquad \sigma^+ = \left[0 \quad 0 \,\middle|\, 0 \quad \frac{K^+}{h^+}\right] a$$

(6)

Terms from interior interface in 1D, $p=1$

$$\tilde{I}_u = -[\![\hat{u}]\!]\,\sigma^* =$$

$$-\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \left(\sum_{\sigma^-} \sigma^- + \sum_{\sigma^+} \sigma^+ + \sum_{v^-} v^- + \sum_{v^+} v^+ \right)$$

$$= -\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \left(\sum_{\sigma^-} \left[0 \quad \frac{K^-}{h^-} \,\middle|\, 0 \quad 0\right] a + \sum_{\sigma^+} \left[0 \quad 0 \,\middle|\, 0 \quad \frac{K^+}{h^+}\right] a + \sum_{v^-} \left[1 \quad 1 \,\middle|\, 0 \quad 0\right] \dot{a} + \sum_{v^+} \left[0 \quad 0 \,\middle|\, 1 \quad 0\right] \dot{a} \right)$$

$$\ldots \sum_{v^-}[1\ 1\ |\ 0\ 0]\, a + \sum_{v^+}[0\ 0\ |\ 1\ 0]\, a \Big)$$

$\tilde{I}_u$ term $\rightarrow$

$$-\underbrace{\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}\left(\sum_{\sigma^-}\left[0\ \ \tfrac{k^-}{h^-}\ \ 0\ \ 0\right] + \sum_{\sigma^+}\left[0\ \ 0\ \ 0\ \ \tfrac{k^+}{h^+}\right]\right)}_{\text{goes to } K^{\sigma}_{4\times4} \text{ interface stiffness}}\, a$$

4×4

$$-\underbrace{\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}\left(\sum_{v^-}[1\ \ 1\ \ 0\ \ 0] + \sum_{v^+}[0\ \ 0\ \ 1\ \ 0]\right)}_{\text{goes to } C^{I}_{4\times4} \text{ interface damping}}\, \dot{a}$$

Other term

$$\tilde{I}_\sigma = \lambda^- \hat{\sigma}^- \cdot n^- \left(-v^* + v^-\right) + \lambda^+ \hat{\sigma}^+ \cdot n^+ \left(-v^* + v^+\right)$$

$$v^* = T_{\sigma^-}\sigma^- + T_{\sigma^+}\sigma^+ + T_{v^-}v^- + T_{v^+}v^+$$