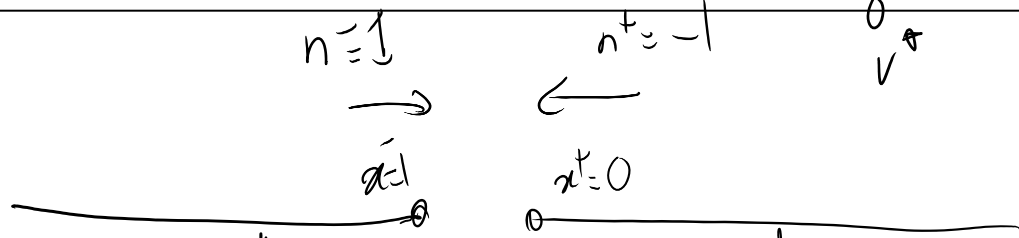
2018/02/28

Wednesday, February 28, 2018 11:37 AM

σ^{ϕ} v^{ϕ}



$a_{4\times 1}$ dofs of the interface

$u^- = [1 \;\; 1 \,|\, 0 \;\; 0]\, a$ $\qquad$ $u^+ = [0 \;\; 0 \,|\, 1 \;\; 0]\, a$

$[[u]] = u^-\cdot n^- + u^+\cdot n^+ = u^- - u^+ = [1 \;\; 1 \,|\, -1 \;\; 0]\, a$

$v^- = [1 \;\; 1 \,|\, 0 \;\; 0]\, \dot{a}$ $\qquad$ $v^+ = [0 \;\; 0 \,|\, 1 \;\; 0]\, \dot{a}$

$\lambda^- \hat{\sigma}^- \cdot \underbrace{n^-}_{1} = \left[0 \;\; \frac{\kappa^-\lambda^-}{h^-} \,\middle|\, 0 \;\; 0\right]$ $\qquad$ $\lambda^+ \hat{\sigma}^+ \cdot \underbrace{n^+}_{-1} = \left[0 \;\; 0 \,\middle|\, 0 \;\; \frac{\lambda^+\kappa^+}{h^+}\right]$

$\lambda^- \hat{\sigma}^- \cdot n^- + \lambda^+ \hat{\sigma}^+ \cdot n^+ = \left[0 \;\; \frac{\kappa^-\lambda^-}{h^-} \,\middle|\, 0 \;\; -\frac{\lambda^+\kappa^+}{h^+}\right]$

$\sigma^- = \left[0 \;\; \frac{\kappa^-}{h^-} \,\middle|\, 0 \;\; 0\right] a$ $\qquad$ $\sigma^+ = \left[0 \;\; 0 \,\middle|\, 0 \;\; \frac{\kappa^+}{h^+}\right] a$

6

Terms from interior interface in 1D, $p=1$

$\tilde{I}_u = -[[\hat{u}]]\, \sigma^{\phi} =$

$$I_u = -\mathbb{I}_u \ldots$$

$$-\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}\left(\sum_{\sigma^-}\sigma^- + \sum_{\sigma^+}\sigma^+ + \sum_{v^-} v^- + \sum_{v^+} v^+\right)$$

$$= -\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}\left(\sum_{\sigma^-}\left[0 \;\; \tfrac{K^-}{h^-} \,\middle|\, 0 \;\; 0\right]a + \sum_{\sigma^+}\left[0 \;\; 0 \,\middle|\, 0 \;\; \tfrac{K^+}{h^+}\right]a + \sum_{v^-}\left[1 \;\; 1 \,\middle|\, 0 \;\; 0\right]\dot{a} + \sum_{v^+}\left[0 \;\; 0 \,\middle|\, 1 \;\; 0\right]\dot{a}\right)$$

$\tilde{I}_u$ term $\rightarrow$

$$\underbrace{-\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}\left(\sum_{\sigma^-}\left[0 \;\; \tfrac{K^-}{h^-} \;\; 0 \;\; 0\right] + \sum_{\sigma^+}\left[0 \;\; 0 \;\; 0 \;\; \tfrac{K^+}{h^+}\right]\right)}_{K_1 \text{ goes to } K^{\sigma}_{4\times4} \text{ interface stiffness}}\, a$$

4×4

$$\underbrace{-\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}\left(\sum_{v^-}\left[1 \;\; 1 \;\; 0 \;\; 0\right] + \sum_{v^+}\left[0 \;\; 0 \;\; 1 \;\; 0\right]\right)}_{C_1 \text{ goes to } C^{I}_{4\times4} \text{ interface damping}}\, \dot{a}$$

Other term

$$\tilde{I}_\sigma = \lambda^- \hat{\sigma}^- . n^- \left(-v^* + v^-\right) + \lambda^+ \hat{\sigma}^+ . n^+ \left(-v^* + v^+\right)$$

$$v^* = \overline{V}_{\sigma^-} \sigma^- + \overline{V}_{\sigma^+} \sigma^+ + \overline{V}_{v^-} v^- + \overline{V}_{v^+} v^+$$

$$= (\lambda \hat{\sigma} n)(-v^*) + \lambda^- \hat{\sigma}^- . n^- v^- + \lambda^+ \hat{\sigma}^+ . n^+ v^+$$

$$(\lambda \hat{\sigma} n) = \lambda^- \hat{\sigma}^- n^- + \lambda^+ \hat{\sigma}^+ n^+$$

$$= -(\lambda \hat{\sigma} n)\Big[\underbrace{V_{\sigma^-} \sigma^- + V_{\sigma^+} \sigma^+}_{K^2} + \underbrace{\overline{V}_{v^-} v^- + \overline{V}_{v^+} v^+}_{C2}\Big]$$

$$+ \underbrace{(\lambda \hat{\sigma} n)^- v^- + (\lambda \hat{\sigma} n)^+ v^+}_{C_2} =$$

$$= (K_2)\, a + C_2\, \dot{a}$$

$$\boxed{\begin{aligned} K_2 &= -(\lambda \hat{\sigma} n)\left(V_{\sigma^-} \begin{bmatrix} 0 \\ \frac{k^-}{h^-} \\ 0 \\ 0 \end{bmatrix}^T + \overline{V}_{\sigma^+} \begin{bmatrix} 0 \\ 0 \\ \hline 0 \\ \frac{k^+}{h^+} \end{bmatrix}^T\right) \\ C_2 &= -(\lambda \sigma n)\left(\overline{V}_{v^-} [1\ 1 \mid 0\ 0] + \overline{V}_{v^+} [0\ 0 \mid 1\ 0]\right) \\ &+ (\lambda \sigma n)^- [1\ 1\ 0\ 0] + (\lambda \sigma n)^+ [0\ 0 \mid 1\ 0] \\ \lambda \hat{\sigma} n^- &= \begin{bmatrix} 0 \\ -\frac{\lambda^- k^-}{h^-} \\ 0 \\ 0 \end{bmatrix} \qquad \lambda \hat{\sigma} n^+ = \begin{bmatrix} 0 \\ 0 \\ \hline 0 \\ \frac{\lambda^+ k^+}{h^+} \end{bmatrix} \qquad \lambda \hat{\sigma} n = \begin{bmatrix} 0 \\ -\frac{\lambda^- k^-}{h^-} \\ 0 \\ \frac{\lambda^+ k^+}{h^+} \end{bmatrix} \end{aligned}}$$

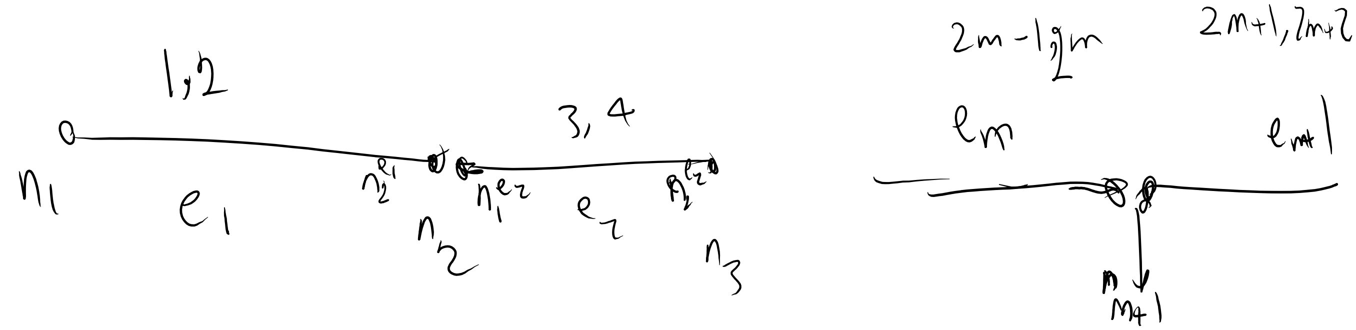
1

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \lambda\sigma n = \begin{bmatrix} 0 \\ \lambda^+ K^+ / h^+ \end{bmatrix} \qquad \lambda\sigma n = \begin{bmatrix} 0 \\ \lambda^+ K^+ / h^+ \end{bmatrix}$$

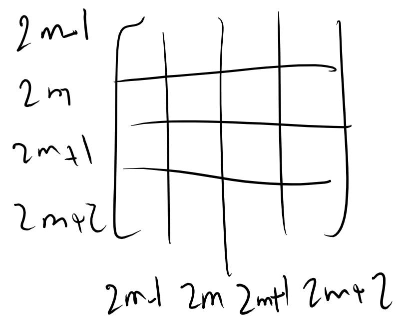
$K^{I} = K_1 + K_2$ 4 x 4 matrix

$C^{I} = C_1 + C_2$ 4 x 4 matrix

we assemble these to the global system



$C^{I} =$



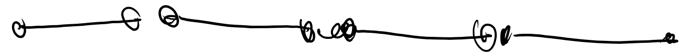
for interface of node m+1

assembled to the global damping matrix C

same process is repeated for K^{I} $\xrightarrow{\text{assembled to}}$ K

same process is repeated for $K^I \xrightarrow{\text{assembled to}} K$

Essential Boundary:

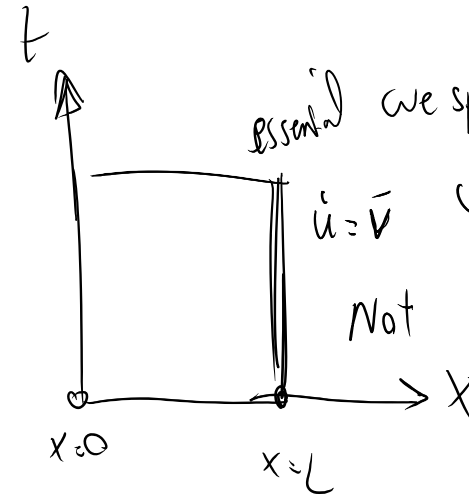


$$B(u,\hat{v}) = -\int_{\partial e} \hat{u}\,(\sigma\cdot n)\,ds + \lambda\int_{\partial e} \hat{\sigma}\cdot n\,(-V^{*}+V)\,ds$$

$$\boxed{\begin{array}{c} V^{*} = \bar{V} \\ \sigma^{*} = \sigma \end{array}}$$

$\dot{u}^{*} = \bar{\dot{u}}$ → essential BC

essential we specify $\dot{u}$ not u



$\dot{u} = \bar{V}$

Not $u = \bar{u}$

if we specify $u(x=L,t) = \bar{u}(t) \Rightarrow$
$V(x=L,t) = \bar{\dot{u}}(t) = \bar{V}(t)$

$$B(u,\hat{v}) = -\int_{\partial e_u} \hat{u}\cdot(\sigma\cdot n)\,ds + \lambda\int_{\partial e_u} \hat{\sigma}\cdot n\,(-\bar{V}+V)\,ds$$

→ 1D

$$\boxed{-\hat{u}\cdot(\sigma\cdot n) + \lambda\,\hat{\sigma}\cdot n\,(-\bar{V}+V) \quad @ \text{ essential BC}}$$

$$-\underbrace{\hat{u}\cdot(\sigma\cdot n)}_{\text{stiffness}} + \lambda\,\hat{\sigma}\cdot n(\underbrace{-\bar{V}}_{\substack{\text{RHS}\\ \text{force}}}+\underbrace{V}_{\text{damping}})$$ @ essential BC

$n=-1$, $x=0$: $u=[1\ 0]c$, $V=[1\ 0]\dot{a}$, $\sigma=[0\ \frac{k}{h}]a$

$x=L$, $n=1$: $u=[1\ 1]c$, $V=[1\ 1]\dot{a}$, $\sigma=[0\ \frac{k}{h}]a$

$K_{\text{contribution}} = -\hat{u}(\sigma\cdot n)$ $\qquad \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & \frac{k}{h}\end{bmatrix} \qquad -\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0 & \frac{k}{h}\end{bmatrix}$

$C_{\text{contribution}} = \lambda\,\hat{\sigma}\cdot n\,V$ $\qquad -\lambda\begin{bmatrix}0\\k/h\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} \qquad \lambda\begin{bmatrix}0\\k/h\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}$

$F(\text{force})_{\text{contribution}} = \lambda\,\hat{\sigma}\cdot n\bar{V}$ $\qquad -\lambda\begin{bmatrix}0\\k/h\end{bmatrix}\bar{V} \qquad \lambda\begin{bmatrix}0\\k/h\end{bmatrix}\bar{V}$

2

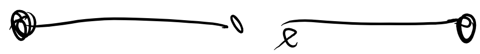
Natural boundary

$$B(\hat{u},u) = -\int_{\partial e_f}\hat{u}\,(\sigma^{*}\cdot n)\,ds + \lambda\int_{\partial e_f}\hat{\sigma}\cdot n\,(-V^{*}+V)\,ds$$

$$\boxed{\begin{aligned}V^{*}&=V\\ \sigma^{*}\cdot n&=\bar{\sigma}\end{aligned}}$$

1D

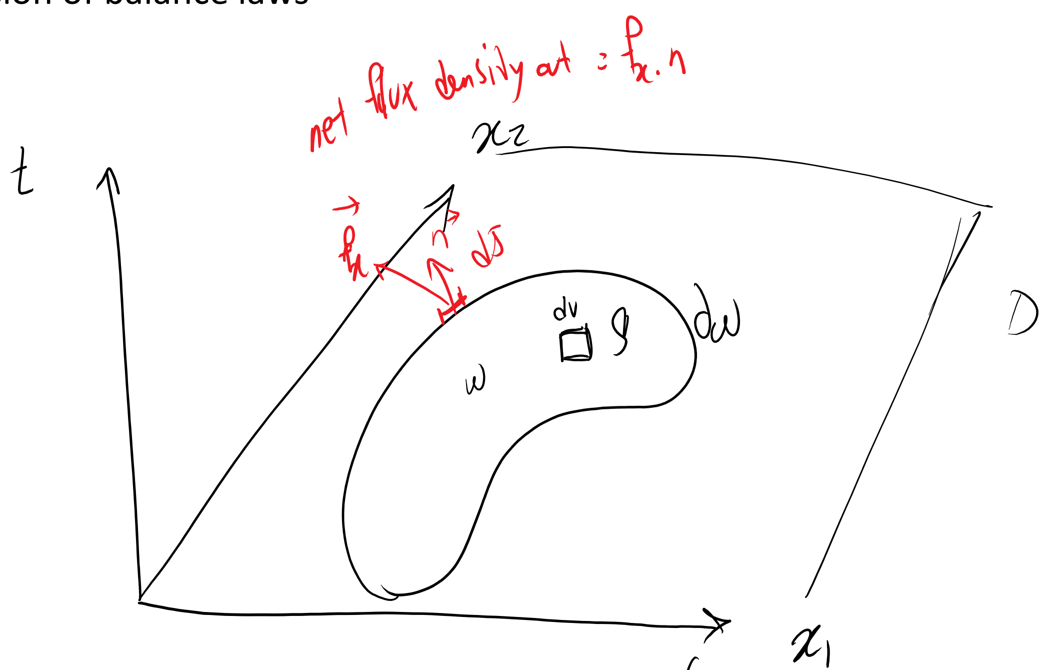
$B(\hat{u}, u) = -\hat{u} \cdot \bar{\sigma}$ only contributes to the force vector



$\hat{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad$ $\hat{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$F = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{\sigma}$ $\qquad$ $F = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{\sigma}$

Spacetime expression of balance laws



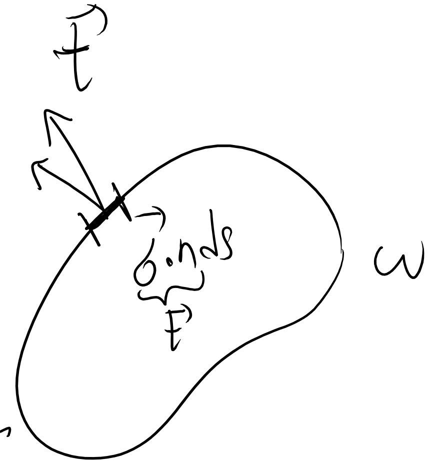
For a static problem $\quad -\int_{\partial\omega} f_x \cdot n \, ds + \int_{\omega} \rho \, dV = 0$

$\underbrace{\partial w}_{\text{net gain of a quantity from } \partial w}$ ... source term or interior contr.

Example:

Solid Mechanics

Force (linear momentum)
static version



because traction ADDs Not subtract forces to w

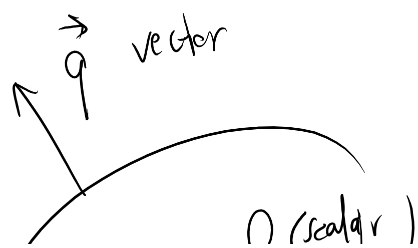
$$-\int_{\partial w} (-\sigma n) ds + \int_{w} \underbrace{(\rho b)}_{S} \cdot dv = 0$$

$$\int_{\partial w} \sigma \cdot n ds + \int_{w} \rho b dv = 0$$

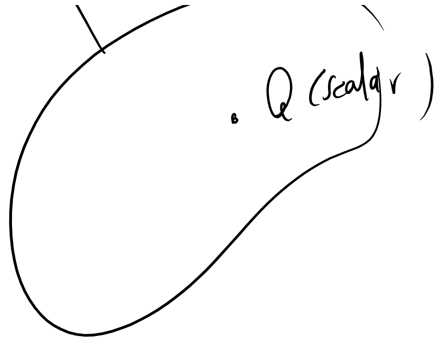
Gauss theorem $\longrightarrow$
$$\left[\begin{array}{l} (\nabla \cdot \sigma + \rho b) \, dv = 0 \quad \text{PDE} \\ \text{on } w \setminus \Gamma \to \text{jump set} \\ [\sigma] \cdot n = 0 \quad \text{Jump condition} \end{array} \right.$$

Another example

$$-\int \vec{q} \cdot n ds + \int Q \, dv = 0$$



$$-\int_{\partial \omega} \vec{q} \cdot n \, ds + \int_{\omega} Q \, dv = 0$$



$$\begin{cases} -\nabla \cdot q + Q = 0 & \text{PDE} \\ [q] \cdot n = 0 & \text{Jump conditions} \end{cases}$$

What if the problem is dynamic?

Example: Heat Conduction

$$\underbrace{-\int_{\partial \omega} q \cdot n \, ds + \int_{\omega} Q \, dv}_{\text{Input energy/time}} \cancel{= 0} = \frac{d}{dt} \mathcal{E}\text{nergy in } \omega$$

= Input power

$$= \frac{d}{dt} \int_{\omega} e \, dv$$

e: energy density per unit volume

$$e = e_{el}(\varepsilon) + \xi_{EM} + \frac{1}{2}\rho v^2 + \cdots$$

$$+ \; e(\cancel{\rho}, T)$$

thermal part

if $e(T)$ is linear

$$C_v T$$

Volumetric heat capacity

$$\boxed{-\int_{\partial\omega} q\cdot n\,ds + \int_{\omega} Q\,dV = \frac{d}{dT}\int_{\omega} (C_v T)\,dV} \quad (3)$$

$$\longrightarrow \quad \boxed{\frac{d}{dt}\underbrace{\int_{\omega} (C_v T)\,dV}_{\text{ext energy}} + \int_{\partial\omega} q\cdot n\,ds = \int_{\omega} Q\,dV}$$

vol. energy density

spatial flux

source law

quantity balanced

Volumetric density of that

Solid Mechanics

$$\frac{D}{Dt} P = \sum \text{Forces}$$

$P =$ mass x velocity

$$\frac{\text{mass x velocity}}{\text{volume}}$$

$$= \left(\frac{\text{mass}}{\text{volume}} \right) \text{x velocity}$$

$$\vec{p} = \rho \vec{V}$$

linear momentum density

$$\frac{d}{dt} P = \frac{d}{dt} \int_{\omega} \rho \, dV = \sum \text{forces} = \int_{\partial\omega} -\sigma n \, ds + \int_{\omega} \rho b \, dV$$

volumetric flux of balanced quantity

outward spatial flux

source term

Balace of Mass in Eulerian framework

$$\frac{\partial}{\partial t} \int_{\omega} \rho \, dv = -\int_{\partial\omega} \rho \vec{v} \cdot n \, ds + 0$$

General form of a balace law:

$$\underbrace{\frac{d}{dt}F}_{\substack{\text{balanced quantity}\\ \text{(energy, mass, linear momentum)}}} = \frac{d}{dt}\int_{\omega} \underbrace{f_t}_{\substack{\text{volumetric}\\ \text{density for } F}}\, dV = -\int_{\partial\omega} \underbrace{f_x \cdot n\, dS}_{\substack{\text{outward}\\ \text{spatial flux}}} + \int_{\omega} \underbrace{S}_{\substack{\text{source}\\ \text{term}}}\, dV \qquad (4)$$

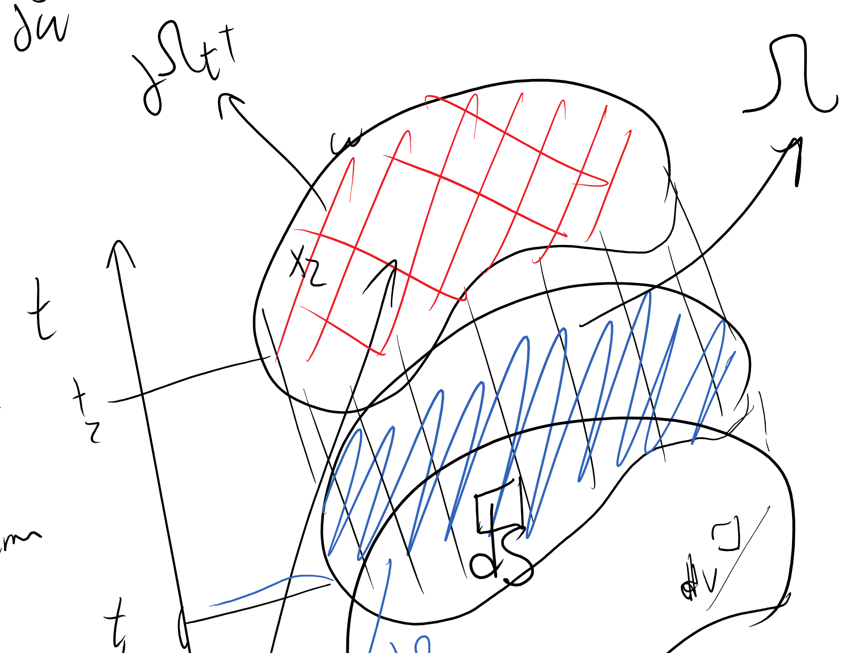
$$(4) \rightarrow \quad \frac{d}{dt}\int_{\omega} f_t\, dV + \int_{\partial\omega} f_x \cdot n\, dS = \int_{\omega} \overset{\nearrow \text{ source term}}{r}\, dV$$

Integrate this over time

$$\int_{t_1}^{t_2}\left(\frac{d}{dt}\int_{\omega} f_t\, dV\right) dt + \int_{t_1}^{t_2} dt \int_{\partial\omega} f_x . n\, dS$$

$$= \int_{t_1}^{t_2} dt \int_{\omega} \underbrace{r}_{\text{source term}}\, dV$$

5.i



$\int_{t_1} \int_{w}$ ↪ sou...

$$\boxed{\int_{t_1}^{t_2} \frac{d}{dt} f \, dt = f(t_2) - f(t_1)}$$

$$\int_{t_1}^{t_2} \left(\frac{d}{dt} \int_{\omega} \rho_t \, dV \right) dt = \int_{\omega} \rho_t \, dV \Big|_{t=t_2} - \int_{\omega} \rho_t \, dV \Big|_{t=t_1}$$

5.ii

$$= \int_{\partial\Omega_t^+} \rho_t \, dS - \int_{\partial\Omega_t^-} \rho_t \, dS$$

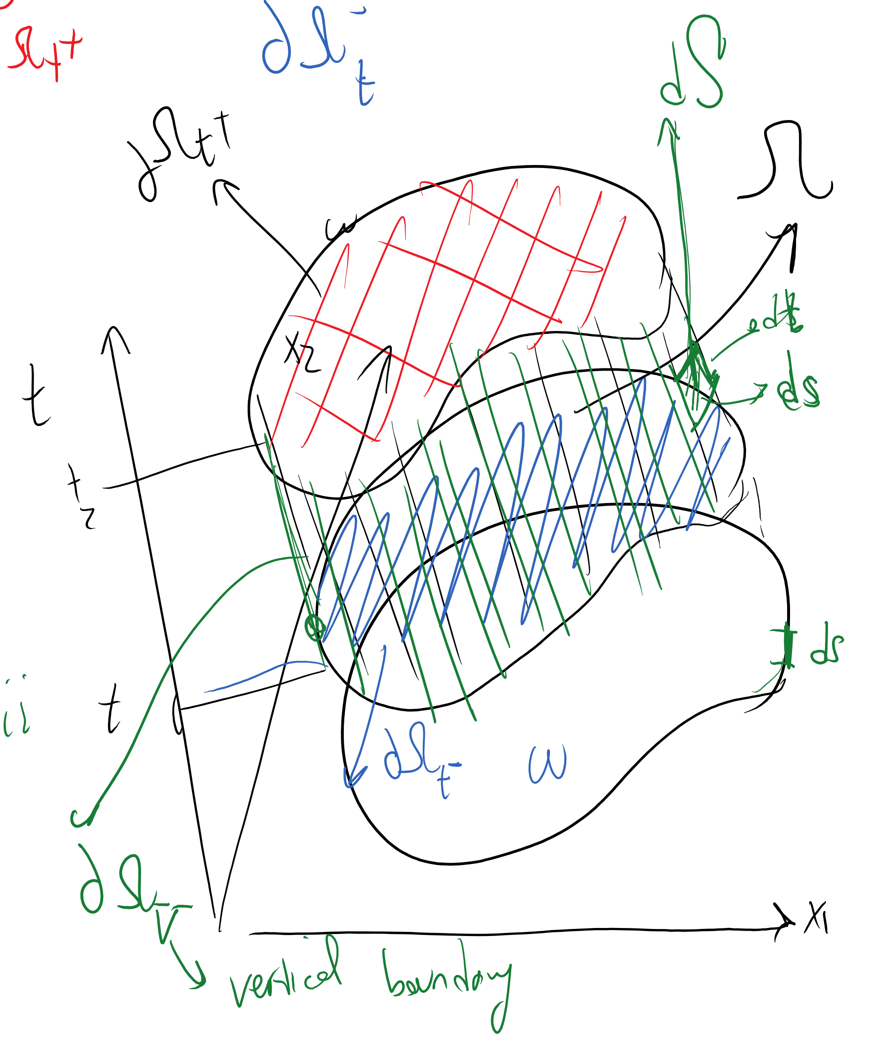
$$\int_{t_1}^{t_2} \left(\int_{\partial w} f_x \cdot n \, ds \right) dt$$

$$= \int_{\partial w} \int_{t_1}^{t_2} (f_x \cdot n) \underbrace{(ds \, dt)}_{dS}$$

$$= \int_{\partial\Omega_V} (f_x \cdot n) \, dS$$

$\partial\Omega_V$ → vertical

5.iii

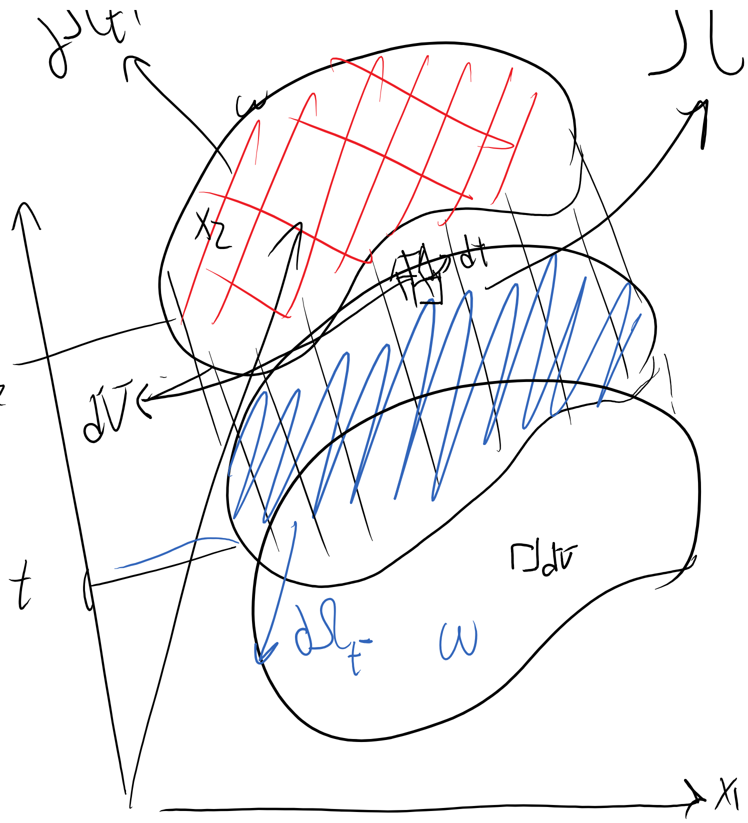


Source term

Source term

$$\int_{t_1}^{t_2} dt \int_{\omega^t} r \, d\bar{V}$$

$$= \int_{\Omega} r \, d\vec{V} \quad \left(\text{Inside } \Omega\right)$$



$$\int_{\partial\Omega_V} (f_x \cdot n) \, dS = \qquad = \vec{N} \text{ on } \partial\Omega_V$$

$$\int \begin{bmatrix} f_x \\ f_t \end{bmatrix} \cdot \begin{bmatrix} \vec{n} \\ 0 \end{bmatrix} dS =$$

$$f_x \cdot \vec{n} + f_t \cdot 0 = f_x \cdot n$$

$$F = \begin{bmatrix} f_x \\ f_t \end{bmatrix}$$

space time flux