From last time the integral over vertical boundary is:
on the top surface we have
\[ \int_{\partial \Omega_4^+} f_t \, dS = \int_{\partial \Omega_4} \left[ \frac{\partial}{\partial t} \right] \cdot [n] \, dS = \int_{\partial \Omega_4} \left[ \frac{\partial}{\partial t} \right] \cdot [n] \, dS \]

Normal vector in space-time on \( \partial \Omega_4^+ \)

\[ \int_{\partial \Omega_4^+} F \cdot N \, dS \]

last boundary integral was on \( \partial \Omega_4 \)

\[ -\int_{\partial \Omega_4^-} f_t \, dS = \int_{\partial \Omega_4} \left[ \frac{\partial}{\partial t} \right] \cdot [n] \, dS \]

\[ F \cdot N \text{ on } \partial \Omega_4^- \]
\[-\int_{\partial T} \mathbf{F} \cdot d\mathbf{S} = \int_{\partial T} \mathbf{F} \cdot d\mathbf{N} ds\]

\[
\int_{\partial B} \mathbf{F} \cdot d\mathbf{N} ds + \int_{\partial B} \mathbf{F} \cdot d\mathbf{N} ds + \int_{\partial B} \mathbf{F} \cdot d\mathbf{N} ds = \int_B d\mathbf{V}
\]

\[
\int_{\partial B} \mathbf{F} \cdot d\mathbf{N} ds = \int_B d\mathbf{V}
\]

\[\text{Expression of balance law in space-time}\]
While we can "justify" the form of balance law in space time (eqn (1)) for an arbitrary shaped $\mathcal{C}$, the actual form of balance law in spacetime is eqn (1).

What I did has many problems! There is no metric in space time; we cannot define normal vectors, etc. and to do all these rigorously we need to use differential forms notation (not discussed here)
For any balance law the contribution of fluxes on the boundary balances the
source term contribution from inside.

Compare this with more complex form of dynamic balance laws are used to:

we often write it as

\[
\frac{d}{dt} \int f \, dv = - \int \mathbf{f} \cdot \mathbf{n} \, ds + \int \mathbf{r} \, dv
\]

not only implied by this \( (\omega \times [t_1, t_2]) \) is more predictive
than general \( \mathbf{f} \) in spacetime, the form looks different from a
static balance law.
Summary:
Any balance law can be written as:

\[ \int f \cdot n \, ds = \int r \cdot d\nu \]

in spacetime if includes (spatial flux density + balance quantity density) & in space \( f \) is only the spatial flux.

Strong form and jump conditions derived from a balance law:

\[ \int f \cdot n \, ds = \int r \cdot d\nu \]

in \( D' \) \( f \in C^1 \) \( f \& \partial f \) are continuous

\[ \int (\nabla \cdot f - r) \, d\nu = 0 \]

\( \omega \) \( C \) \( N' \) is arbitrary
\[ \int_{\Omega} (\nabla \cdot f - \mathbf{r}) \, dv = 0 \quad \text{\( \omega \subseteq \Omega \) is arbitrary} \]

Localizability theorem \[\rightarrow\] strong form \[4a\] (PDE)

\[ \forall \mathbf{x} \in D \setminus \Gamma \]

wherever \( \mathbf{r} \in C^1 \)

Jump conditions

\[ \int_{\partial \Omega^1} ds = \int_{\partial \Omega^2} ds \quad (W_1) \]

\[ \int_{\partial \Omega^1} ds = \int_{\partial \Omega^2} ds \quad (W_2) \]

\[ \int_{\partial \Omega^1} ds = \int_{\partial \Omega^2} ds \quad (W_3) \]

and then

\[ \int_{\partial \Omega^1} ds + \int_{\partial \Omega^2} ds + \int_{\partial \Omega^1} ds = \int_{\partial \Omega^2} ds + \int_{\partial \Omega^1} ds \]

\[ \int_{\partial \Omega^1} ds + \int_{\partial \Omega^2} ds + \int_{\partial \Omega^1} ds = \int_{\partial \Omega^2} ds + \int_{\partial \Omega^1} ds \]

\[ \int_{\partial \Omega^1} ds + \int_{\partial \Omega^2} ds + \int_{\partial \Omega^1} ds = \int_{\partial \Omega^2} ds + \int_{\partial \Omega^1} ds \]

\[ \int_{\partial \Omega^1} ds + \int_{\partial \Omega^2} ds + \int_{\partial \Omega^1} ds = \int_{\partial \Omega^2} ds + \int_{\partial \Omega^1} ds \]

\[ \int_{\partial \Omega^1} ds + \int_{\partial \Omega^2} ds + \int_{\partial \Omega^1} ds = \int_{\partial \Omega^2} ds + \int_{\partial \Omega^1} ds \]
\[ \int_{\partial \Omega} f \cdot n \, ds + \int_{\Gamma} (f^+ \cdot n + f^- \cdot n) \, ds = \int_{\Omega} \text{div} \, \mathbf{a} \, dx \]

\[ \int_{\partial \Omega} f \cdot n \, ds = 0 \], balance law for \( \omega_3 \)

4. \[ \int_{\partial \Omega} (f^+ \cdot n + f^- \cdot n) \, ds = 0 \]

\[ f^+ \cdot n + f^- \cdot n = 0 \]

\( \forall x \in \bar{\Omega} \) by localization

\[ f^- \cdot n + f^+ \cdot n = \llbracket f \rrbracket \]
\[ f \cdot n + f^{-} \cdot n = 0 \]

Another view point

\[ f \cdot n + f^{-} \cdot n = 0 \]

\[ f \cdot n + f^{-} \cdot (-n) = 0 \]

\[ -(f_{+} - f_{-}) \cdot n = 0 \]

\[ [f] = f_{\text{outside}} - f_{\text{inside}} \]

\[ f_{n}^{-} + f_{n}^{+} = 0 \Rightarrow [f] \cdot n = 0 \]
We must start from the balance law. Many times the PDEs are written in a non-conservative way. For example Navier-Stokes equations may be written by dividing certain equations by \( \rho \). In that case, we cannot even "guess" what the jump conditions corresponding to the original balance laws would have looked like.

Examples:

**Heat Conduction**

**Static version**
Basic version

\[(q^+ - q^-) \cdot n = 0\]

\[\{q\} \cdot n = 0 \quad \{q\} \perp n\]

\[q^+ \cdot n = -q^- \cdot n\] balance of net forces

PDE \[r = \mathbf{a} \quad f = q \quad \nabla \cdot q - \mathbf{a} = 0\]

Solid Mechanics

\[f = -d \quad r = \rho b\]

PDE \[\nabla \cdot (b) - \rho b = 0 \quad \nabla \cdot d + \rho b = 0\]

\[(\delta^+ - \delta^-) \cdot n = 0\]

\[\delta^+ \cdot n - \delta^- \cdot n = 0\]

\[\delta^+ \cdot n = \delta^- \cdot n\] action - reaction law
Traction are equal, but do we need to have

\[ [g] = 0 \]

\[ \partial \neq \partial \]

\[ \partial \neq \partial \]

How about a dynamic balance law:

\[ F = \int [\text{linear momentum}] \]

\[ f_x = -\partial \]

\[ f_t = \text{linear momentum} \]

\[ P = \text{linear momentum} = M \gamma \]
\[ F = \begin{bmatrix} -\delta \\ pV \end{bmatrix} \]

For balance of linear moment:

\[ N = \begin{bmatrix} n_x \\ n_t \end{bmatrix} \]

\[ [F] \cdot N = 0 \]

\[ \begin{bmatrix} -\delta \\ pV \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_t \end{bmatrix} = 0 \Rightarrow -[\delta]n_x + [p]n_t = 0 \]

- \[ -[\delta]n_x + [p]n_t = 0 \]

a) \[ n_t = 0 \]

- \[ -[\delta]n_x = 0 \]

Vertical "time-like"

Action-reaction

\[ N = \begin{bmatrix} n_x \\ n_t \end{bmatrix} \]

\[ \text{radii are (not) equal} \]

\[ \text{slidi equa} \]
Example of $N$ being slant
\[ \eta = \frac{1}{\sqrt{1 - c^2}} \]
\[ \eta_t = \frac{c}{\sqrt{1 - c^2}} \]

\[ [\mathbf{a}] \cdot n_x \neq 0 \quad ? \]

\[ [\mathbf{a}] \cdot n_x + [\mathbf{p}] \cdot n_t = 0 \quad ? \quad \text{check it} \]

Where (what direction) can these jumps happen?

on vertical manifolds one can have nontrivial

\[ N = \begin{bmatrix} n_x \\ 0 \end{bmatrix} \]
can have nontrivial jumps

\[ [-6] \cdot n_x + [P] \cdot 0 = 0 \]

\[-[B] \cdot n_x = 0 \rightarrow \text{tractions are continuous}\]

\([P] \neq 0 \text{ is not possible}\]

Other nontrivial jumps

\[ C = \sqrt{\frac{E}{P}} \]

See dynamic example above