# 2018/03/05

Monday, March 05, 2018 11:38 AM

## From last time

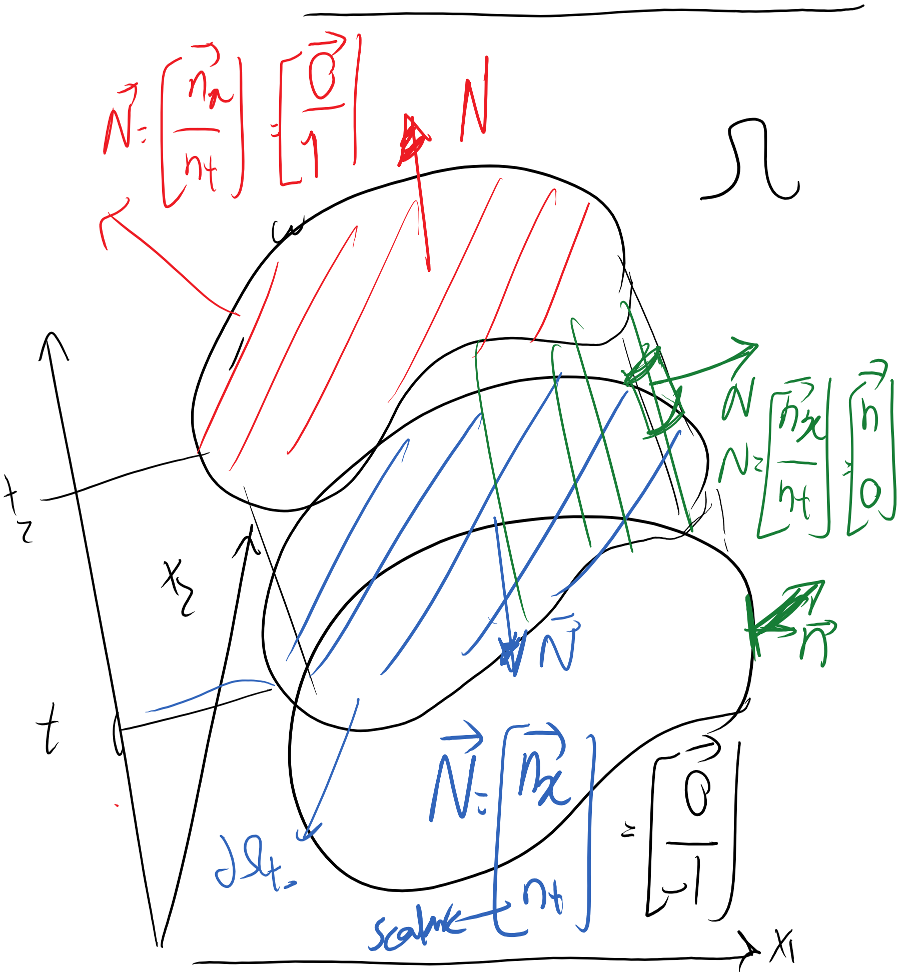
$$\int_{\partial\Omega_v} (f_x \cdot n)\, dS = \qquad = \vec{N} \text{ on } \partial\Omega_v$$

$$\int \begin{bmatrix} f_x \\ f_t \end{bmatrix} \cdot \begin{bmatrix} \vec{n} \\ 0 \end{bmatrix} dS =$$

$$f_x \cdot \vec{n} + f_t \cdot 0 = f_x \cdot n$$

$$F = \begin{bmatrix} f_x \\ f_t \end{bmatrix}$$

space time flux




## From last time the integral over vertical boundary is:

$$\boxed{\int_{\partial\Omega_v} f_x \cdot n\, dS = \int_{\partial\Omega_v} F \cdot N\, dS}$$

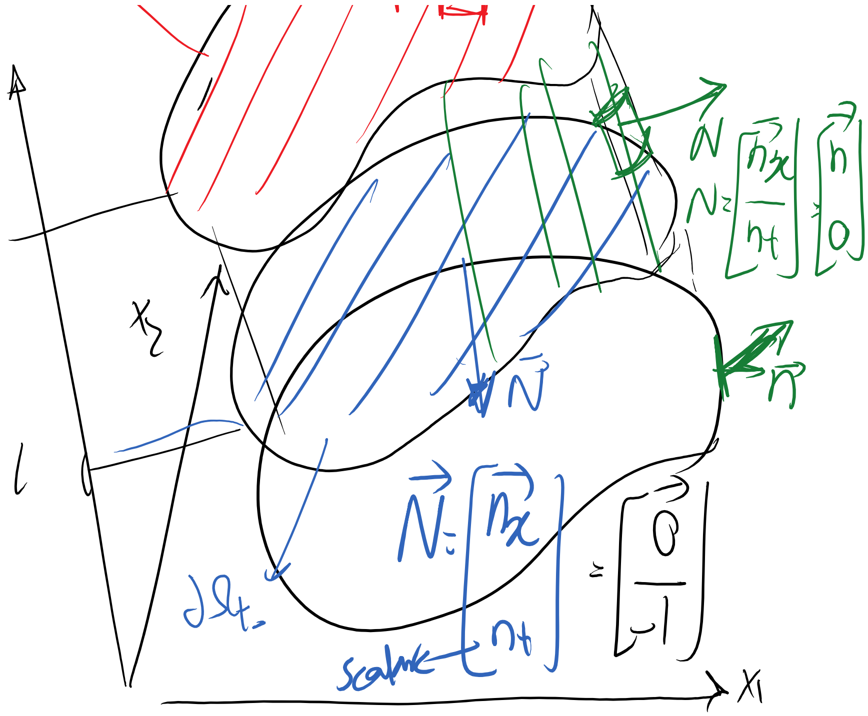
i

on the top surface we have

$$\int_{\partial\Omega_{t+}} f_t \, dS$$

$$= \int \overbrace{\underbrace{\begin{bmatrix} f_x \\ f_t \end{bmatrix} \cdot \begin{bmatrix} \vec{0} \\ 1 \end{bmatrix}}_{f_x \cdot \vec{0} + f_t \cdot 1 = f_t}}^{F} \, dS$$

$\begin{bmatrix} \vec{0} \\ 1 \end{bmatrix}$ → Normal vector in spacetime on $\partial\Omega_{t^+}$




$$= \int_{\partial\Omega_{t^+}} F \cdot N \, dS$$

$$\boxed{\int_{\partial\Omega_{t^+}} f_t \, dS = \int_{\partial\Omega_{t^+}} F \cdot N \, dS} \quad !!$$

last boundary integral was on $\partial\Omega_{t^-}$

$$-\int_{\partial\Omega_{t^-}} f_t \, dS = \int_{\partial\Omega_{t^-}} \underbrace{\begin{bmatrix} f_x \\ f_t \end{bmatrix}}_{F} \cdot \underbrace{\begin{bmatrix} \vec{0} \\ -1 \end{bmatrix}}_{\vec{N} \text{ on } \partial\Omega_{t^-}} \, dS$$

$$\boxed{-\int_{\partial\Omega_t^-} f_t \, dS = \int F \cdot N \, dS} \quad \text{iii}$$

$$\int_{\partial\Omega_V} f_x \cdot n \, dS + \int_{\partial\Omega_t^+} f_t \, dS - \int_{\partial\Omega_t^-} f_t \, dS = \int_{\Omega} r \, dV$$

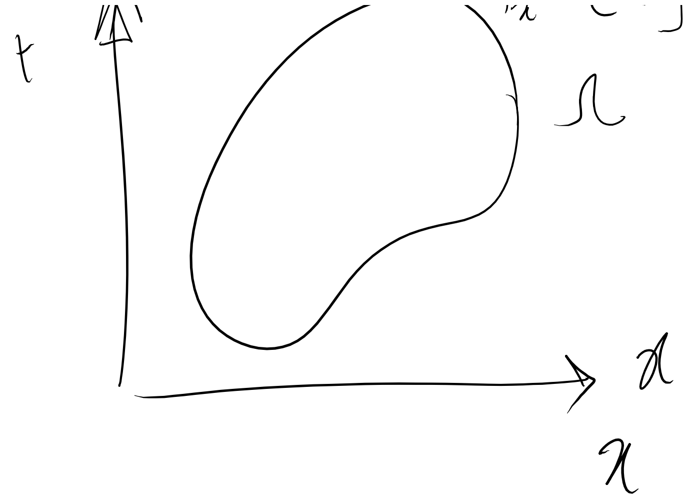
$$\Big\downarrow \text{i}$$

$$\int_{\partial\Omega_V} F \cdot N \, dS + \int_{\partial\Omega_t^+} F \cdot N \, dS + \int_{\partial\Omega_t^-} F \cdot N \, dS = \int_{\Omega} r \, dV$$

$$\text{(1)} \quad \boxed{\int_{\partial\Omega} F \cdot N \, dS = \int_{\Omega} r \, dV}$$

Expression of balance law in space time

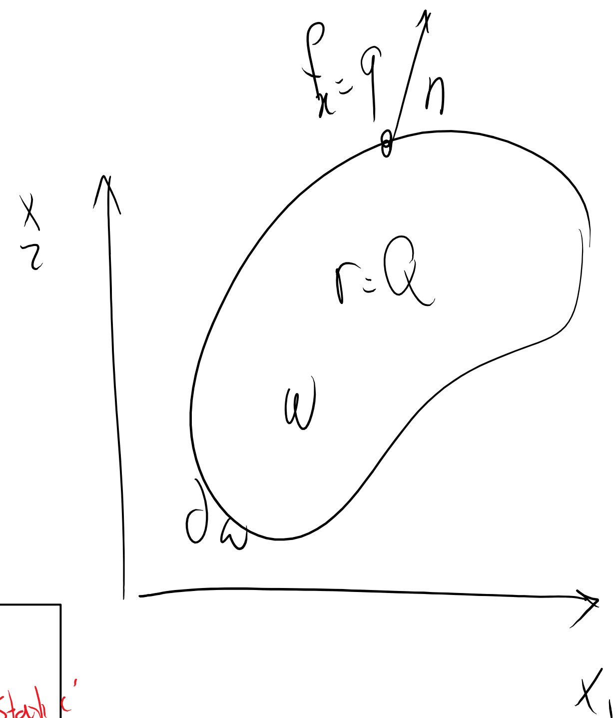

$$\vec{N} = \begin{bmatrix} n_x \\ n_t \end{bmatrix}$$




While we can "justify" the form of balance law in space time (eqn (1)) for an arbitrary shaped $\Omega$ , the actual form of balance law in spacetime is eqn (1).

What I did has many problems! There is no metric in space time; we cannot define normal vectors, *etc.* and to do all these rigorously we need to use differential forms notation (not discussed here)

Comparison with steady state or static balance laws



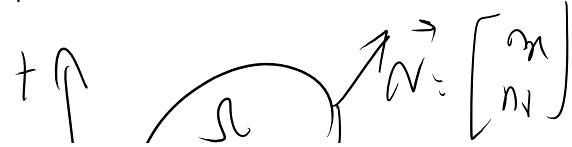

$$-\int_{\partial w} f_x \cdot \vec{n}\, ds + \int_w r\, dV = 0$$

2a

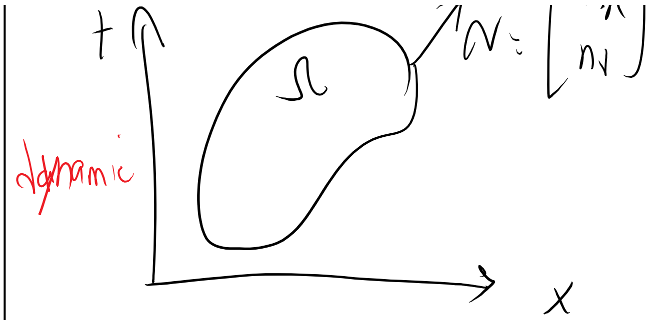
$$\int_{\partial w} f_x \cdot n_x\, ds = \int_w r\, dV$$

Static

Space time balance law



$$\boxed{\int_{\Omega} F \cdot N \, dS = \int_{\Omega} r \, d\bar{V}} \quad \text{2b}$$



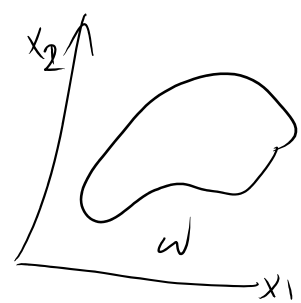

$$F = \begin{bmatrix} F_x \\ F_t \end{bmatrix}$$

For any balance law the contribution of fluxes on the boundary balances the source term contribution from inside.

Compare this with more complex form of dynamic balance laws are used to:

we often write 2b as

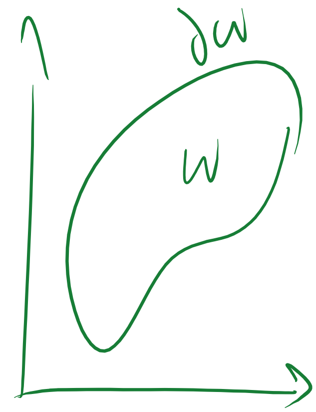
$$\frac{d}{dt} \int_{\omega} f_t \, dV = - \int_{\partial \omega} f_x \cdot n \, dS + \int_{\omega} r \, dV$$




not only Ω implied by this ($\omega \times [t_1 \; t_2]$) is more restrictive than general Ω in spacetime, the form looks different from a static balance law.

Summary:
Any balance law can be written as:

(3) $$\boxed{\int_{\partial\omega} f\cdot n \, ds = \int_{\omega} r \, dv}$$

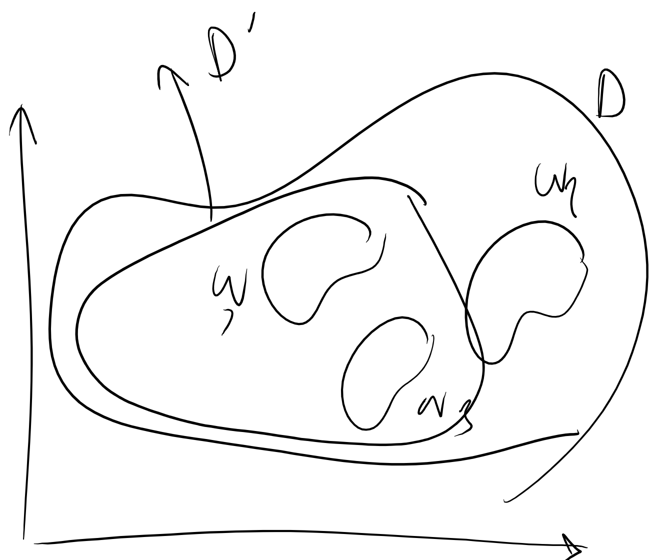



in spacetime $f$ includes (spatial flux density + balance quantity density) & in space $f$ is only the spatial flux.

Strong form and jump conditions derived from a balance law:

$\forall \omega$

$$\int_{\partial\omega} f\cdot n \, ds = \int_{\omega} r \, dv$$




in $D'$ $f \in C^1$ $f$ & $\nabla f$ are continuous

$$\int (\nabla\cdot f - r) \, dv = 0 \qquad \omega \subset D' \text{ is arbitrary}$$

$$\int_{\omega} (\nabla \cdot f - r)\, dv = 0 \qquad \omega \subseteq D' \text{ is arbitrary}$$

Localization theorem $\rightarrow$

strong form

(PDE)

4a

$$\boxed{\nabla \cdot f - r = 0 \quad \forall x \in D \setminus \Gamma}$$

jump manifolds

where ever $f \in C^1$

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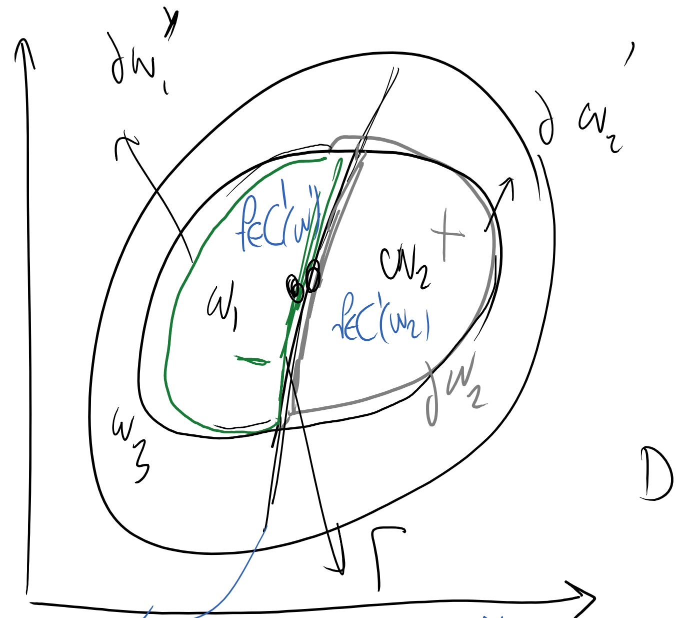
Jump conditions

$$\int_{\partial \omega_1} f \cdot n \, ds = \int_{\omega_1} r \, ds \qquad (\omega_1)$$

$$\int_{\partial \omega_2} f \cdot n \, ds = \int_{\omega_2} r \, ds \qquad (\omega_2)$$




add them

a manifold on which $f$ may not be $C^1$

$$\int_{\partial \omega_1'} f \cdot n \, ds + \int_{\Gamma} f^- \cdot n \, ds + \int f \cdot n \, ds + \int f^+ n^+ \, ds = \int r \, dv + \int r \, dv$$

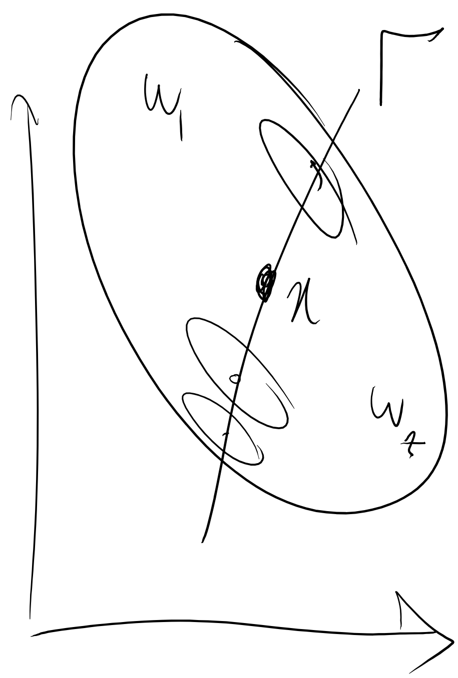
$$\int_{\partial w_2} f\cdot n\, ds + \int_{\Gamma} f^{+} n^{+}\, ds = \int_{w_1} r\, dV + \int_{w_2} r\, dV$$

$$\int_{\partial w_3} f\cdot n\, ds + \int_{\Gamma} \left( f^{-}\cdot n^{-} + f^{+}\cdot n^{+} \right) ds = \int_{w_3} r\, dV$$

$$\Rightarrow \quad \int_{\partial w_3} f\cdot n\, ds - \int_{w_3} r\, dV = 0 .$$

balance law for $\omega_3$

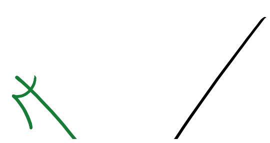
④ $$\boxed{\int_{\Gamma} \left( f^{-}\cdot n^{-} + f^{+}\cdot n^{+} \right) ds = 0}$$



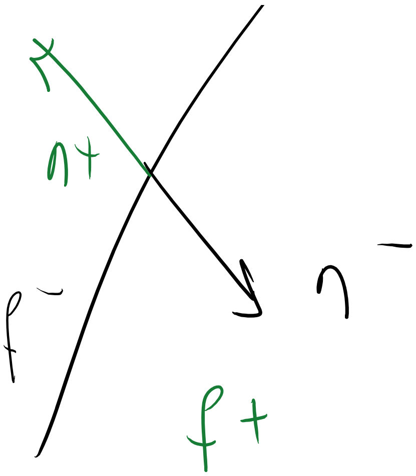

$$\Rightarrow \quad f^{-}\cdot n^{-} + f^{+}\cdot n^{+} = 0$$

$\forall x \in \Gamma$ by localization

$$f^{-}\cdot n^{-} + f^{+}\cdot n^{+} = [\![ f ]\!]$$



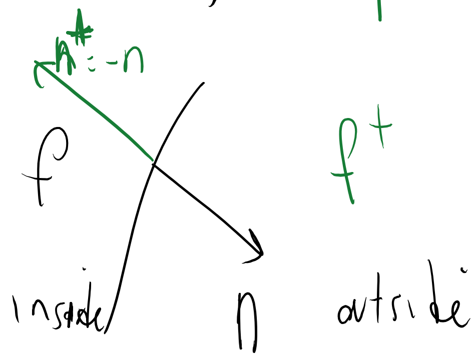
$f \cdot n + f \cdot n = [\![ \; | \; ]\!]$




symmetric view point

Jump def. by Arnold 2000 2001 papers

Another view point




$$f^- \cdot n^- + f^+ \cdot n^+ =$$

$$f \cdot n + f^+ \cdot (-n) =$$

$$-\left(f^+ - f^-\right) \cdot n = 0$$

$$[f] = f^{\text{outside}} - f^{\text{inside}}$$

$$f^- n^- + f^+ n^+ = 0 \longrightarrow [f] \cdot n = 0$$

Summary

$$\int_{\partial \omega} f \cdot ds = \int_{\omega} r \, dv \implies$$

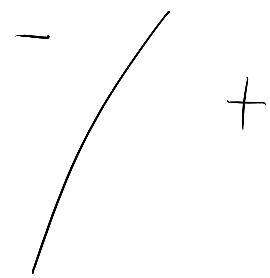
1a. PDE / Strong form $\quad \nabla \cdot f - r = 0 \quad D \setminus J$

$J \downarrow$ Jump set

2. Jump condition

$$f^- \cdot n^- + f^+ \cdot n^+ = 0$$

$[[f]] = 0$ Arnold's notation




or

$[f] \cdot n = 0 \qquad [f] = f^{out} - f^{in}$

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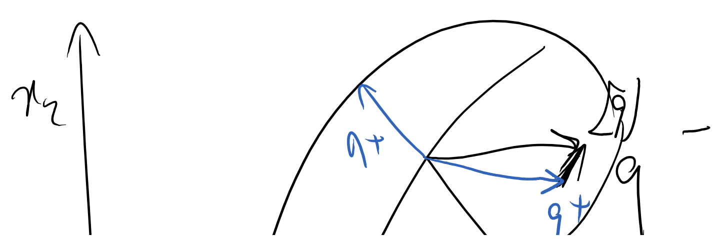
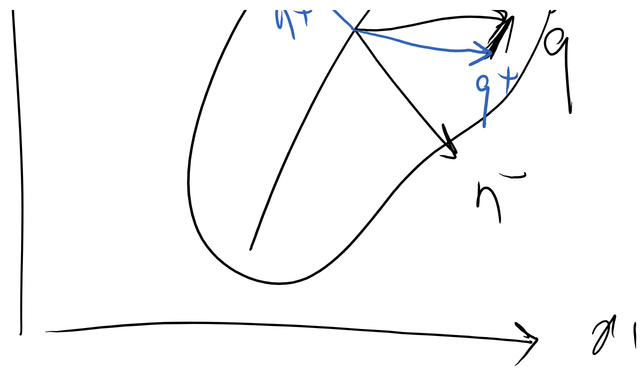
We must start from the balance law. Many times the PDEs are written in a non-conservative way. For example Navier-Stokes equations may be written by dividing certain equations by \rho. In that case, we cannot even "guess" what the jump conditions corresponding to the original balance laws would have looked like.

Examples: Heat conduction

static version



Static version




$$(q^+ - q^-) \cdot n = 0$$

$$[q] \cdot \vec{n} = 0 \qquad [q] \perp n$$

$$q^+ \cdot n^+ = -q^- \cdot n^-$$ balace of net fluxes

PDE $\quad r = Q \quad f = q \quad \nabla \cdot q - Q = 0$

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Solid Mechanics


$$f = -\sigma \qquad r = \rho b$$

PDE $\nabla \cdot (-\sigma) - \rho b = 0 \qquad \nabla \cdot \sigma + \rho b = 0$

$$(\sigma^+ - \sigma^-) \cdot n = 0$$

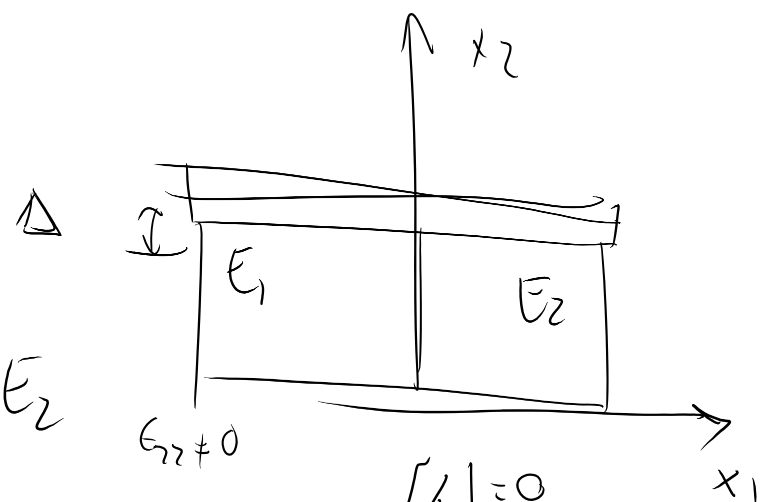
$$\sigma^+ \cdot n - \sigma^- \cdot n = 0 \qquad \boxed{\sigma^+ \cdot n = \sigma^- \cdot n}$$

action - reaction law

$\sigma^+ \cdot n^+ = -\sigma^- \cdot n^-$

Tractions are equal, but do we need to have

$$[\sigma] = 0$$




$E_1 \neq E_2$

$\sigma_{22} \neq 0$

$[\sigma_{11}] = 0$

$[\sigma_{12}] = 0$

$[\sigma_{22}] \neq 0$

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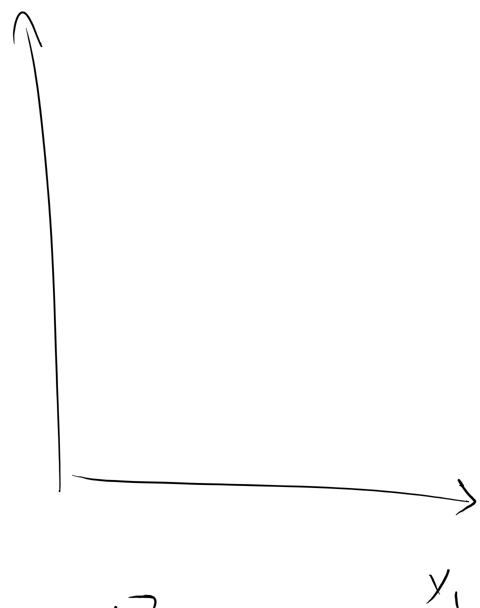
How about a dynamic balance law:

$$F = \begin{bmatrix} f_x \\ f_t \end{bmatrix}$$

balance of linear momentum




$f_x = -\sigma$

$f_t = \dfrac{\text{Linear momentum}}{\text{volume}}$ P = Linear momentum $= M\vec{v}$

volume

mass velocity

$$= \frac{\text{mass} \cdot \vec{v}}{\text{volume}} = \rho \vec{V}$$

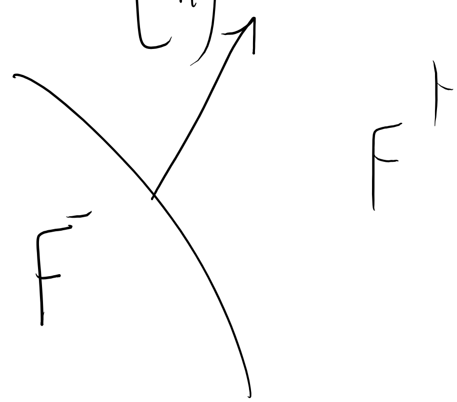
$\rho$

$$f_t = p := \rho \vec{V}$$

$$\boxed{F = \begin{bmatrix} -\sigma \\ \rho \vec{V} \end{bmatrix}}$$

6 for balance of linear momentum

$$N = \begin{bmatrix} n_x \\ n_t \end{bmatrix}$$




$$[F] \cdot N = 0$$

$$\begin{bmatrix} [-\sigma] \\ [\rho V] \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_t \end{bmatrix} = 0 \rightarrow -[\sigma] n_x + [p] n_t = 0$$

$$-[\sigma] n_x + [p] n_t = 0$$


a) $n_t = 0 \quad -[\sigma] \cdot n_x = 0$

action-reaction

tractions are (-) equal

static equation

$$N = \begin{bmatrix} n_x \\ 0 \end{bmatrix}$$

"static equati

b) $n_x = 0$

$(\rho v)^+$ $N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

horizontal "space-like" $(\rho v)^-$

$$[p] \cdot 1 = [p] = 0$$

$$(\rho v)^- - (\rho v)^+ = 0$$

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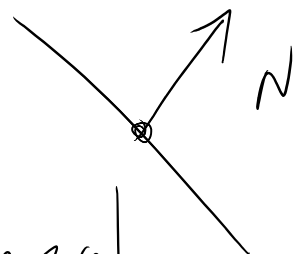
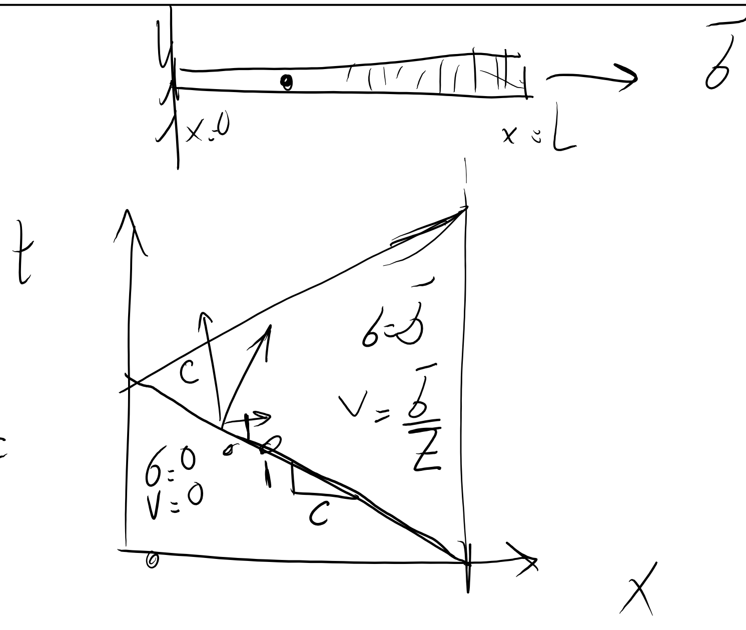
c)

$$-[\sigma] n_x + [p] n_t = 0$$




$[\sigma] n_x$ & $[p] n_t$ both can be nonzero!

Example of N being slant




$c = \sqrt{\frac{E}{\rho}}$

$n_x = \frac{1}{\sqrt{1+c^2}}$

$n_t = \frac{c}{\sqrt{1+c^2}}$

$Z = c\rho = \sqrt{E\rho}$

$p = \rho v$

$[\sigma]\cdot n$

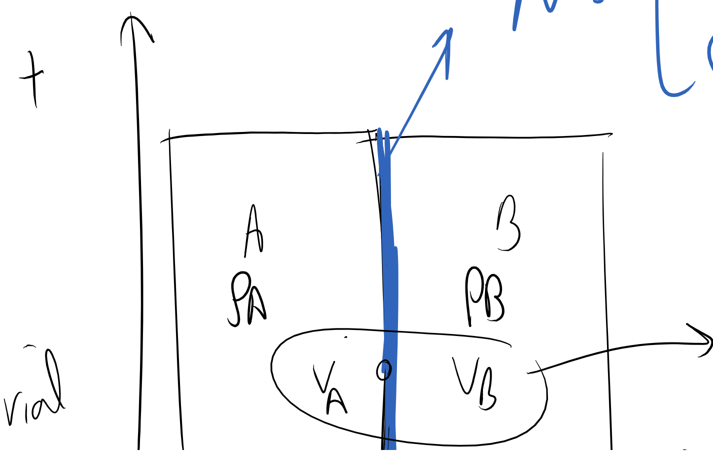
$[\sigma]\cdot n_x \neq 0$ ?

$[-\sigma]\cdot n_x + [p]\cdot n_t = 0$ ? check it

where (what direction) can these jumps happen?

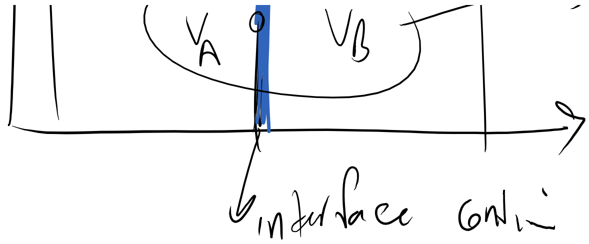
$N = \begin{bmatrix} n_x \\ 0 \end{bmatrix}$

on vertical manifolds we can have nontrivial




simple case bonded interface

can have nontrivial jumps




bonded interface

$[v]=0$

$[p]\neq 0$ ($\rho_A \neq \rho_B$)

$p=\rho v$

$$[-\sigma]\cdot n_x + [p]\cdot 0 = 0$$

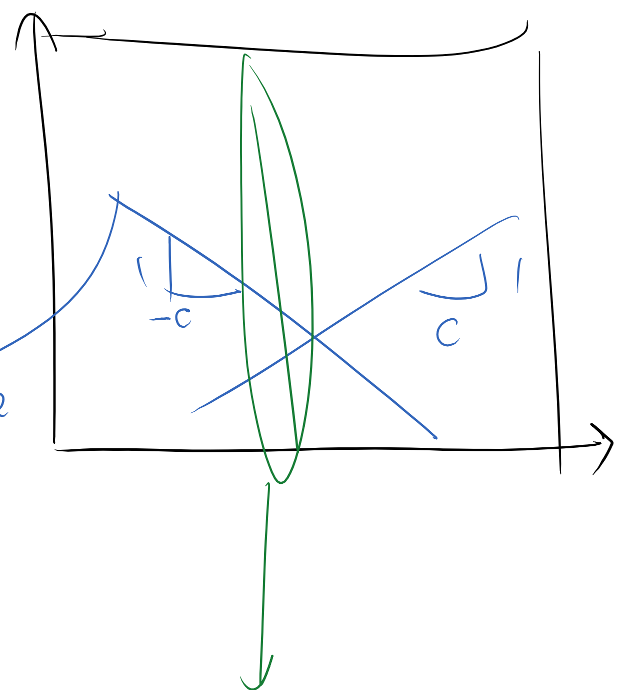
$$-[\sigma]\cdot n_x = 0 \rightarrow$$ tractions are continuous

$[p]\neq 0$ is it possible

Other nontrivial jumps

$$c = \sqrt{\frac{E}{\rho}}$$

see dynamic example above




No (no nontrivial jumps)