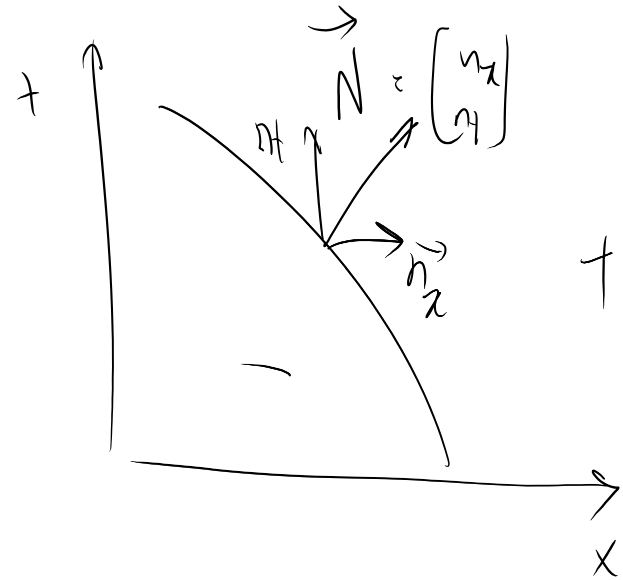
2018/03/07

Wednesday, March 07, 2018 11:38 AM

$$F = \begin{bmatrix} f_x \\ f_t \end{bmatrix}$$

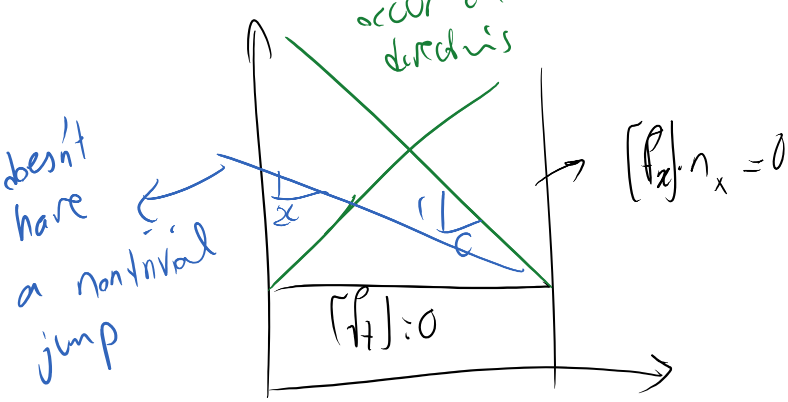


$$[F]\cdot N = 0$$

$$(F^+ - F^-)\cdot N = 0$$

in fluid mechanics this is called Rankine-Hugoniot Jump condition

non trivial jumps occur on characteristic directions

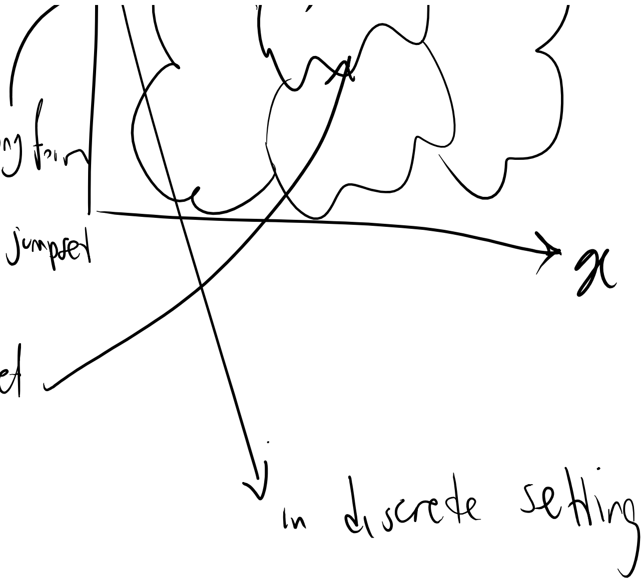


How a these conditions used in formulating a space time DG method

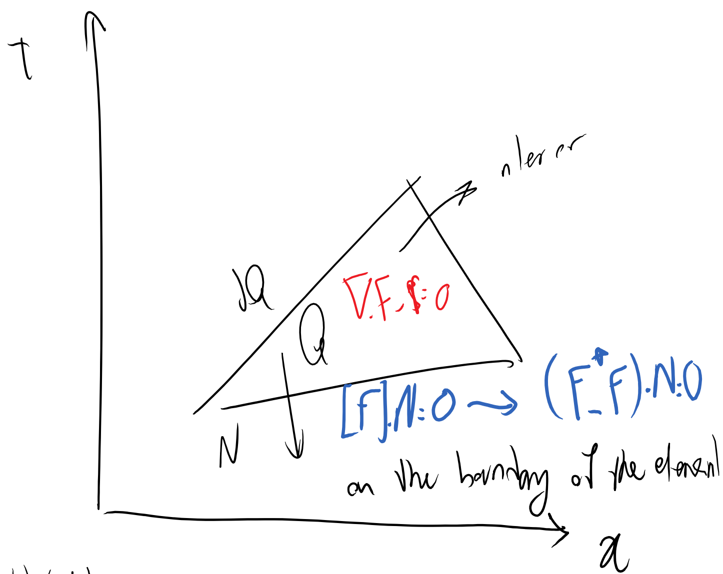
$$\int_{\partial\Omega} F\cdot N \, dS = \int_{\Omega} r \, dV = 0 \iff$$

$\partial\Omega$ Ω

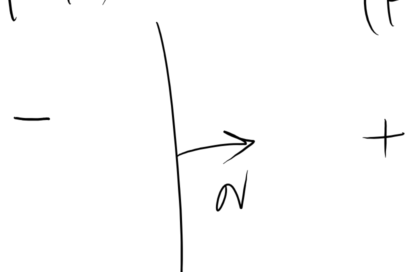
$$\begin{cases} \nabla\cdot F - r = 0 & \text{PDE/strong form} \quad \Omega\setminus J_N \ \text{jumpset} \\ [F]\cdot N = 0 & \text{on jumpset} \end{cases}$$



In discrete setting we satisfy these equations inside & on the boundary of finite elements.



$(F^+ - F^-)\cdot N = 0$ ⊕ $\qquad (F^- - F^+)\cdot(-N) = 0$



⊖

it's basically one condition

$(F^* - F^-)\cdot N = 0, \qquad (F^* - F^+)\cdot(-N) = 0$

(b) $\quad (F^* - F^-).N = 0 \quad | \quad (F^* - F^+).(-N) = 0$

adding them $\quad (F^+ - F^-).N = 0$
which is (a)

So, (b) used in the context of FV and DG methods is more flexible than simply writing (F+ - F-).N = 0. It provides many options for the definition of numerical flux (for example, average, Riemann and various forms of approximate Riemann fluxes)

WRS:

discrete space of solutions inside an element, e.g P_3 order polynomials

$\forall Q, \; \forall w \in V_Q$

elements in space time

$$\int_Q w . R_i \, dV + \int_{\partial Q} w . R_b \, dS = 0 \qquad (1)$$

$R_i = \nabla . F - r \qquad R_b = (F^* - F).N$

Weak statement

$$\int_Q w(\nabla\cdot F) = \int_Q \nabla\cdot(wF) - \nabla w\cdot F = \int_{\partial Q} wF - \int_Q \nabla w F$$

plug this into (1) we get

$$\boxed{\forall Q, \forall w \in V_Q : \int_Q \left(-\nabla w\cdot F - w\, r\right) dV + \int_{\partial Q} w F^{*} dS = 0}$$ (2)

weak statement

Interpolation of solution:

Remember if time marching schemes we had:

$$T(x,t) = \sum_i T_i(x)\, a_i(t) \longrightarrow C\dot{a} + Ka = F$$

semi-discrete problem

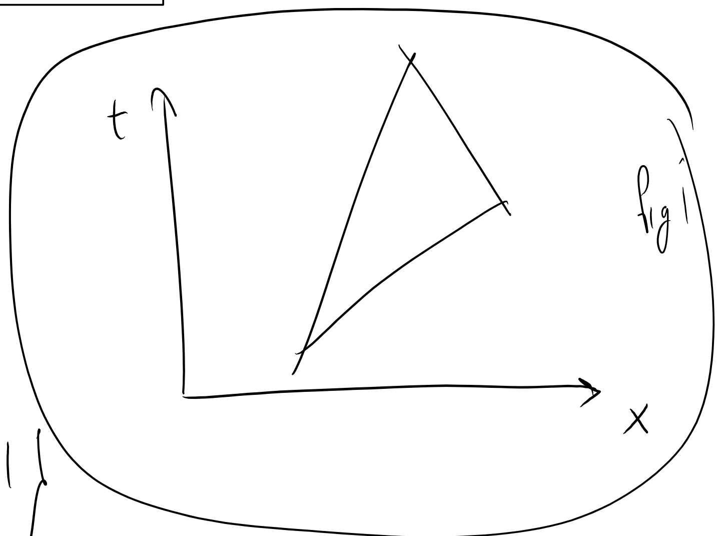
How about Now

(3) $$\boxed{T(x,t) = \sum_i T_i(\vec{x},t)\, a_i} \longrightarrow \underbrace{K}_{\text{stiffness matrix}} a = f$$

③ $\boxed{T(x,t) = \sum_i T_i(x,t)\, a_i}$ $\longrightarrow$ $\underbrace{K}_{\text{system stiffness matrix}} a = T$

fully space time formulation

Examples



$p=0$ $\quad T_i$'s $= \{1\}$

$p=1$

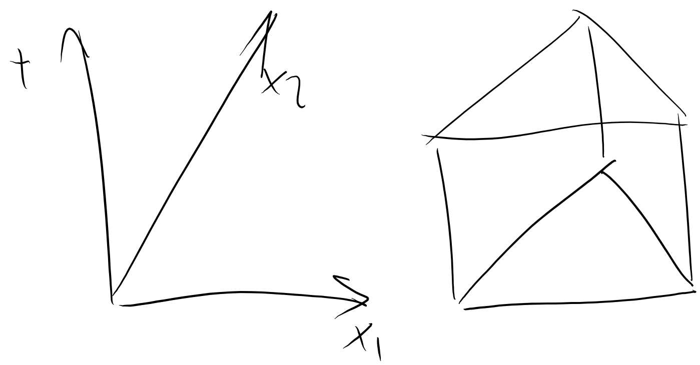
$$\begin{array}{ccc} & 1 & \\ x & & t \end{array}$$

$p=2$

$$\begin{array}{ccccc} & & 1 & & \\ & x & & t & \\ x^2 & & xt & & t^2 \end{array}$$

$\vdots$

Ex. 2



$p = 2$ in space

$p = 1$ in time

basis in space

$$\begin{array}{ccc} & 1 & \\ x_1 & & x_2 \\ x_1^2 & x_1x_2 & x_2^2 \end{array} \times \begin{bmatrix} 1 \\ t \end{bmatrix} \text{ in time}$$

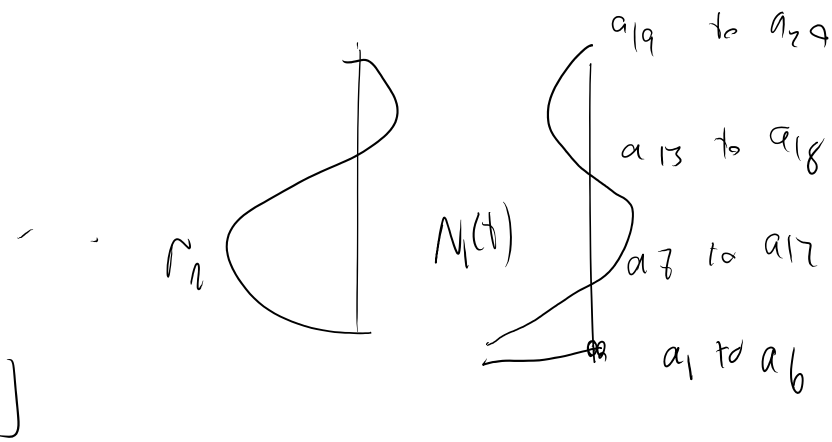
$(1, x_1, \ldots, x_2^2), t, x_1 t, \ldots, x_2^2 t$

12 terms

what if $p_t = 3$ $\begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$ can be used

for $p_x = 2$ Tri in space

Or something like



$[N_1(t), \ldots N_4(t)]$

In fact, by using shape functions like this and integrating the balance law in spacetime and getting rid of time dependencies we get something **similar** to RK4 implicit integration in time. There are many examples in literature that time integration schemes are derived this way for a particular problem.

We'll focus more on fig. 1 type of elements.

We'll do two sample formulation for the thermal heat conduction and elastodynamics.

General balance law form $\int_{\partial\Omega} F \cdot N \, dS - \int_{\Omega} r \, dV = 0 \longrightarrow$

weak statement

$$\boxed{\int_{\Omega} \left(\nabla w \cdot F - w r \right) dV + \int_{\partial\Omega} w F^{*} N \, dS = 0}$$

How do we get balance law property satisfied per element in DG methods

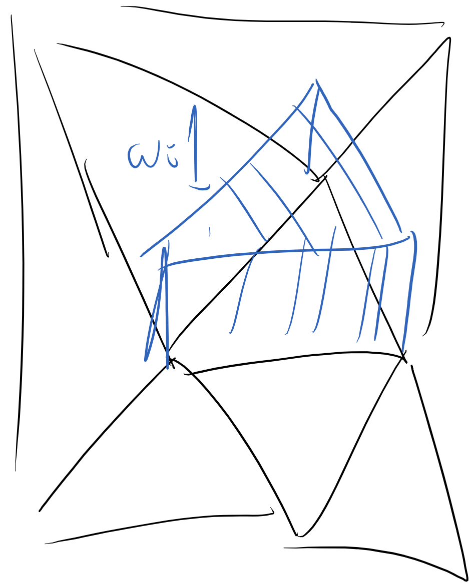
Lets use $\underline{w = 1}$

(2) $\longrightarrow$ $\int \left(\cancel{\nabla 1} \cdot F - 1 r \right) dV + \int 1 \, F^{*} N \, dS = 0$

② $\rightarrow$ $$\int_\Omega \left(\cancel{\nabla} 1 \cdot f - 1\, r\right) dV + \int_{\partial\Omega} 1 \cdot f^{\partial} \cdot N \, dS = 0$$

④ $$\boxed{\int_{\partial\Omega} f^{\partial} \cdot N \, dS - \int_\Omega r \, dV = 0}$$

Balance property per element for DG methods



We cannot get this element level balance property in general for CFEMs because the weight function cannot be set to 1 in one element only (but in DG we can). There is also the issue of fluxes on the boundary of elements …

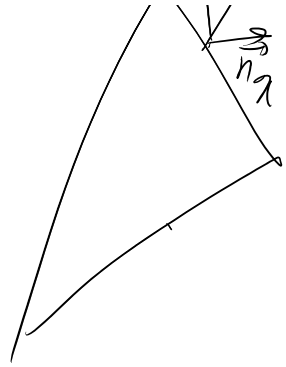
In CFEMs we can get the balance property for the whole domain (where w = 1 can be set)

Expansion of the weak form

$$\int_\Omega \left(\nabla_{st} w \cdot f - w\, r\right) dV + \int_{\partial\Omega} w f^{\partial} \cdot N \, dS = 0$$

$\nabla w = \left[\nabla w \right.$ $\rightarrow$ spatial part

$$\nabla_{st} w = \begin{bmatrix} \nabla w \\ \mathring{w} \end{bmatrix} \xrightarrow{} \text{temporal part}$$



$$F = \begin{bmatrix} f_x \\ f_t \end{bmatrix} \qquad F^{\circledast} = \begin{bmatrix} f_x^{\circledast} \\ f_t^{\circledast} \end{bmatrix} \quad N = \begin{bmatrix} n_x \\ n_t \end{bmatrix}$$

$$\int_Q \left(\begin{bmatrix} \nabla w \\ \dot{w} \end{bmatrix} \cdot \begin{bmatrix} f_x \\ f_t \end{bmatrix} - wr \right) dV + \int_{\partial Q} w \begin{bmatrix} f_x^{\circledast} \\ f_t^{\circledast} \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_t \end{bmatrix} dS = 0$$

(5) $$\boxed{\int_Q \left(-\nabla w\, f_x - \dot{w}\, f_t - wr \right) dV + \int_{\partial Q} w \left(f_x^{\circledast} n_x + f_t^{\circledast} n_t \right) dS = 0}$$

Expanded weak statement

Example 1: Thermal problem

$$\frac{d}{dt} \int_{\omega} \underbrace{CT}_{f_t}\, dV = - \int_{\partial\omega} \underbrace{q \cdot n}_{f_x}\, ds + \int_{\omega} \underbrace{\cancel{Q}}_{r}\, dV$$

heat conduction

"balance of energy"

$$F = \begin{bmatrix} q \\ CT \end{bmatrix} \begin{matrix} \rightarrow f_x \\ \rightarrow f_t \end{matrix} \qquad r = \cancel{Q}$$

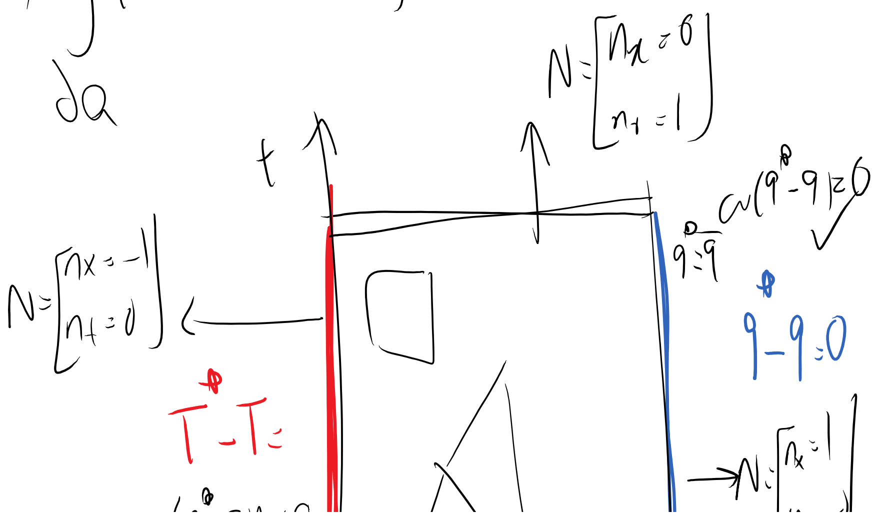
strong form: $\nabla_{st} \cdot F - r = 0 \Rightarrow \underbrace{\nabla \cdot q + (CT)}_{\nabla_{st} \cdot F} - r = 0$

For heat eqn weighted residual & weak form look like

$$\boxed{\begin{aligned} &\text{WR} \quad \int_\Omega w \cdot (\nabla \cdot F - r)\, dV + \int_{\partial\Omega} w (F^* - F) \cdot N\, dS = 0 \rightarrow \int_\Omega w (C\dot{T} + \nabla \cdot q - Q) + \int_{\partial\Omega} w\big((q^* - q) n_x + (CT^* - CT) n_t\big)\, dS = 0 \\ &\text{WF} \quad \int_\Omega (-\nabla w \cdot F - wr)\, dV + \int_{\partial\Omega} w F^* \cdot N\, dS = 0 \qquad \int_\Omega (-\dot{w} CT - \nabla w \cdot q - wQ) + \int_{\partial\Omega} w (q^* n_x + CT^* n_t)\, dS = 0 \end{aligned}}$$

WRS:

$$\int_\Omega w (C\dot{T} + \nabla \cdot q - Q) + \int_{\partial\Omega} w\big((q^* - q) n_x + (CT^* - CT) n_t\big)\, dS = 0$$



$T = \bar{T}(t)$

$T = \bar{T}(t)$ ———— $q = \bar{q}(t)$

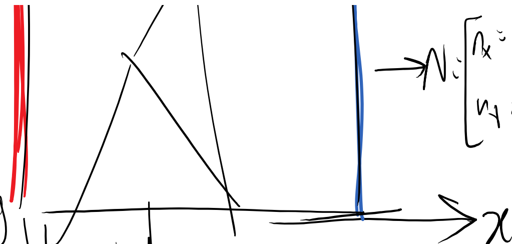
Essential

$T - T =$

$w(q^* - q)n_x = 0$

$q^* = q$

this is trivially satisfied!



$N: \begin{bmatrix} n_x = 1 \\ n_t = 0 \end{bmatrix}$

$N: \begin{bmatrix} n_x = 0 \\ n_t = -1 \end{bmatrix}$

$CT^* - CT = 0$

IC

$$R_b = w(q^* - q)\, n_x + (CT^* - CT)\, n_t$$

stabilizer for parabolic/elliptic PDEs

Solution $q^* = \{q\} + \alpha [[T]]$ (Arnold, Cockburn, ...)

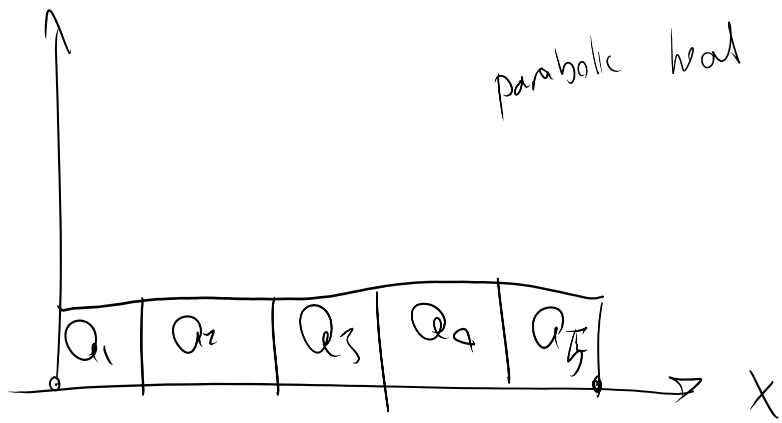
so this added term will indirectly enforce $T - T$ on essential BC

Another approach is adding a jump term to boundary integral as we did before for the solution of parabolic/elliptic PDEs

$$\boxed{\text{WRS} \quad \int \overset{\hat{T}}{w} \cdot (C\dot{T} + \nabla \cdot q - r)\, dV + \int_{\partial\Omega} \overset{\hat{T}}{w} \left((q^* - q)\, n_x + (T^* - T)\, n_t \right) dS + \int_{\partial\Omega} \hat{q} \cdot n_x (T^* - T)\, dS = 0}$$

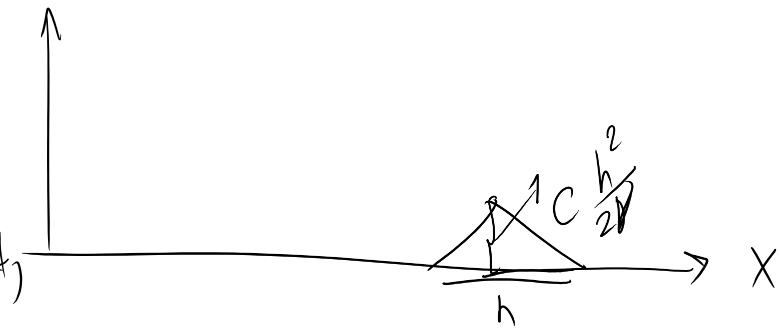
this way we can enforce T^* on essential BC parabolic heat conduction

implicit solution of the problem in space



explicit time

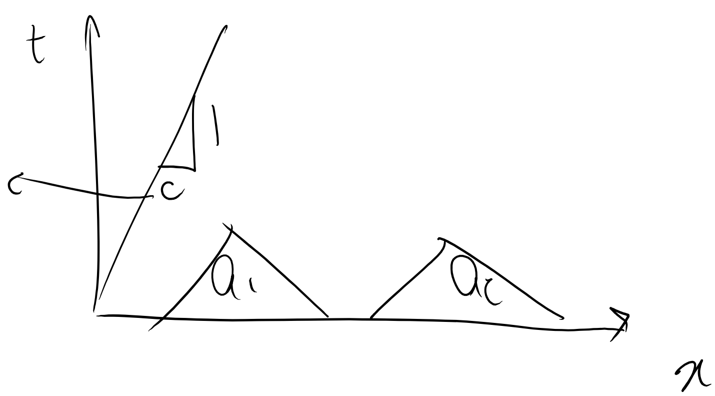
needs stability proof



How about a hyperbolic heat conduction model?

In this case there is a maximum wave speed.

wave speed



There are many different hyperbolic models for heat conduction. One is the MCV model