2018/03/19

Monday, March 19, 2018 11:41 AM

Fourier heat law

$$q = -k\nabla T \quad \rightarrow$$

PDE

$$C\dot{T} + \nabla \cdot q = Q$$

$$C\dot{T} - \nabla \cdot k \nabla T = Q$$

parabolic

1D

$$C\dot{T} - \frac{\partial}{\partial x} k \frac{\partial T}{\partial x} = Q$$

PDE in two variables

$$A u_{,xx} + B u_{,xy} + C u_{,yy} + D u_{,x} + E u_{,y} + F u = g$$

$$\left(\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \underbrace{\begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \right) u + D u_{,x} + E u_{,y} + F u = g$$

$$\det \mathcal{A} = AC - \frac{B^2}{4} \begin{cases} < 0 & \text{hyperbolic} \\ > 0 & \text{elliptic} \\ = 0 & \text{parabolic} \end{cases}$$

Higher # dep arguments

$$\begin{bmatrix} C_{x_1x_1} & & \text{sym} \\ & C_{x_ix_i} & \\ & & C_{x_nx_n} \end{bmatrix}$$

we need to look at the eigen values of A to classify PDE type

$$C\dot{u} - \overset{\text{constant}}{K}\, u_{,xx} = 0$$

t like y

$$\underbrace{-K}_{A}\, u_{,xx} + \underbrace{0}_{B}\, u_{,xt} + \underbrace{0}_{C}\, u_{,tt} + \dots = 0$$

$$AC - \left(\frac{B}{2}\right)^2 = 0 \quad \rightarrow \text{parabolic}$$

Another example 1D elastodynamics

$$\sigma_{,x} - p_{,t} = 0$$

$\sigma = E\varepsilon = E u_{,x}$

$p = \rho v = \rho \dot{u}$

assume E, ρ constant

$$E u_{,xx} - \rho u_{,tt} = 0$$

$$E\, u_{,xx} + 0\, u_{,xt} + (-\rho)\, u_{,tt} = 0$$

$\downarrow \qquad \downarrow \qquad \downarrow$

$A \qquad B \qquad C$

$AC - \left(\frac{B}{2}\right)^2 = -\epsilon\rho < 0$ hyperbolic

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We are going to solve a hyperbolized version of heat conduction:

$$\left.\begin{aligned} C\dot{T} + \nabla\cdot q &= Q \\ q &= -\kappa\nabla T \end{aligned}\right] \rightarrow \left\{\begin{aligned} C\dot{T} + \nabla\cdot q &= Q \\ \Sigma\dot{q} + \nabla\cdot(\underbrace{\kappa}_{\text{conductivity matrix}} T) &= -q + \nabla\cdot\kappa\, T \end{aligned}\right. \quad \text{eqn } (1)$$

$$\nabla\cdot\kappa T = (\kappa_{ij}T)_{,j} = \kappa_{ij,j}T + \kappa_{ij}T_{,j} = \nabla\cdot\kappa\, T + \kappa\nabla T$$

$\boxed{\kappa\nabla T = \nabla\cdot\kappa T - \nabla\cdot\kappa\, T}$ used in (1)

$$\dot{f}_t + \nabla\cdot f_x = S \qquad \text{system of conservation laws}$$

MCV heat eqn.

$$\begin{aligned} C\dot{T} + \nabla\cdot q &= Q \\ \Sigma\dot{q} + \nabla\cdot\kappa T &= -q + \nabla\cdot\kappa\, T \end{aligned} \qquad (2)$$

$$f_t = \begin{bmatrix} CT \\ \tau q \end{bmatrix}$$ scalar, vector

$$f_x = \begin{bmatrix} q \\ \kappa T \end{bmatrix}$$ vector, 2nd order tensor, conductivity matrix

$$S = \begin{bmatrix} Q \\ -q + \nabla \cdot \kappa T \end{bmatrix}$$

$$\boxed{\begin{gathered} \int_e \omega \left(\underbrace{\dot{f}_t + \nabla \cdot f_x - S}_{} \right) dv + \int_{\partial e} \omega \left[\left(f_t^* - f_t \right) n_t + \left(f_x^* - f_x \right) n_x \right] ds = 0 \\ \omega \quad \nabla_{st} \cdot F \qquad\qquad \omega \left(F^* - F \right) \cdot N \\ F = \begin{bmatrix} f_x \\ f_t \end{bmatrix} \end{gathered}}$$

$$f_t = \begin{bmatrix} CT \\ \tau q \end{bmatrix} \qquad \omega = \begin{bmatrix} \hat{T} \\ \hat{q} \end{bmatrix} \qquad f_x = \begin{bmatrix} q \\ \kappa T \end{bmatrix}$$

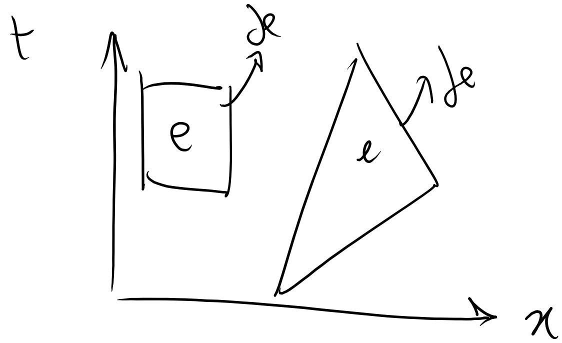
$$\int_e \begin{bmatrix} \hat{T} \\ \hat{q} \end{bmatrix} \cdot \left(\begin{pmatrix} C\dot{T} + \nabla \cdot q - Q \\ \tau \dot{q} + \kappa \nabla T + q \end{pmatrix} \right) dv + \int_{\partial e} \begin{bmatrix} \hat{T} \\ \hat{q} \end{bmatrix} \begin{pmatrix} (CT^* - CT) n_t + (q^* - q) \cdot n_x \\ (\tau q^* - \tau q) n_t + (\kappa T^* - \kappa T) n_x \end{pmatrix} ds = 0$$

$\Rightarrow$

$\Rightarrow$

(3)

$$\boxed{\begin{aligned}&\int_e \hat{T}\left(C\dot{T} + \nabla\cdot q - Q\right)dv + \int_{\partial e} \hat{T}\left[\left(\underbrace{CT^{*}}_{IC} - CT\right)n_t + \left(q^{*} - q\right)n_x\right]ds \; + \\ &\int \hat{q}\left(\tau\dot{q} + \kappa\nabla T + q\right)dv + \int \hat{q}\left[\left(\underbrace{\tau q^{*}}_{IC} - \tau q\right)n_t + \left(\kappa T^{*} - \kappa T\right)n_x\right]ds = 0\end{aligned}}$$

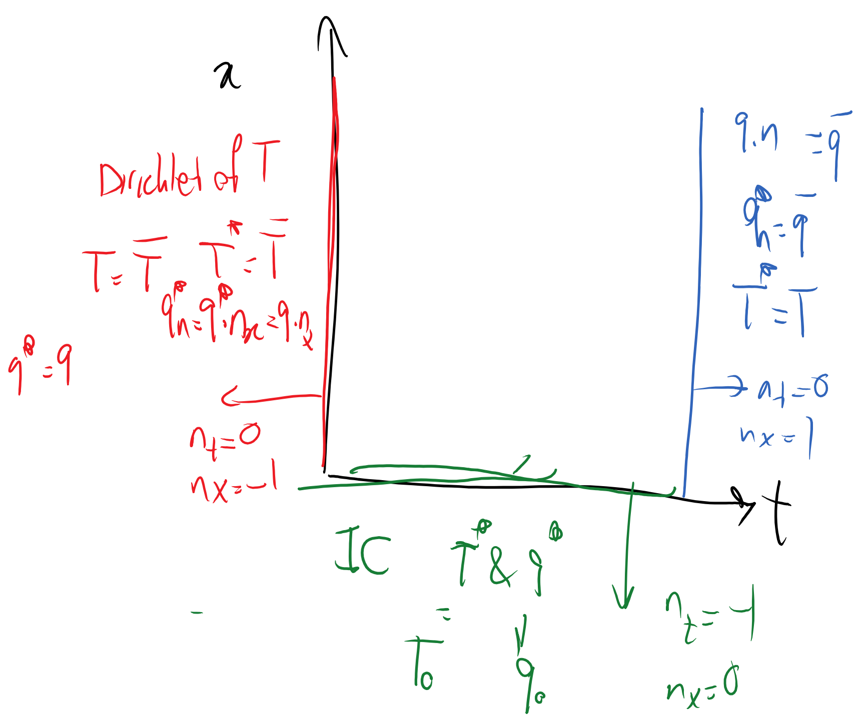


primary fields solved for the problem are T & q

$q_1^{*} = q^{*}\cdot n_x$

& T^{*}

must be specified



in parabolic eqn only T^{*} was specified

BCs:

Last time for parabolic heat conduction equation, only q*.nx appeared in the WRS:

$$wf \int_{\Omega} w\cdot(\nabla\cdot F - s)\,dV + \int_{\partial\Omega} w(F^* - F)\cdot N\,dS = 0 \;\rightarrow\; \int_{\Omega} w(c\dot{T} + \nabla\cdot q - Q) + \int_{\partial\Omega} w\left[(q^* - q)n_x + (cT^* - cT)n_t\right] dS = 0$$

$$w\,kf \int_{\Omega} (-\nabla w\, F - ws)\,dV + \int_{\partial\Omega} w F^*\cdot N\,dS = 0 \qquad \int_{\Omega} (-\dot{w}cT - \nabla w\cdot q - wQ) + \int w(q^* n_x + cT^* n_t)\,dS = 0$$

Now we have both q*nx and T* (so essential BC on T can be specified)

In (3) to get back to Fourier heat model we need to set the relaxation time equal to zero.

$\tau = 0$

$$\int_{e} \hat{T}(c\dot{T} + \nabla\cdot q - Q)\,dv + \int_{\partial e} \hat{T}\left[(\underbrace{cT^*}_{IC} - cT)n_t + (q^* - q)n_x\right] ds \;+$$

$$\int \hat{q}(\cancel{\tau\dot{q}} + \kappa\nabla T + q)\,dv + \int \hat{q}\left[\cancel{(\underbrace{\tau q^*}_{IC} - \tau q)n_t} + (\kappa T^* - \kappa T)n_x\right] ds = 0$$

$$\int_e \hat{T}\left(c\dot{T} + \nabla\cdot q - Q\right) dV + \int_{\partial e} \hat{T}\left((c\underline{T}^* - cT)n_t + (q^* - q) n_x \right) ds$$

$$+ \int_e \hat{q}\left(\underbrace{\kappa \nabla T + q}_{\text{compatibility / const law eqn}}\right) dV + \int_{\partial e} \hat{q}\left(\kappa T^* - \kappa T\right) n_x \, ds = 0 \quad (4)$$

2 - field formulation for Fourier heat law (T, q are interpolated)

satisfied weakly in 2F formulation

enables us to enforce T^* on ∂D_u

Do we have any problems specifying T^* & $q^* n_x$ on essential & natural BC's respectively?

This reduction from a hyperbolic PDE is another approach to get to the additional weight term that has (T* - T), see the formulation at the end of the last session for another approach.

To reduce this to a 1-field formulation we interpolate T only and get rid of compatibility / const. equation term

(5)

$$\int_e \hat{T}\left(C\dot{T} + \nabla\cdot q - Q\right) dV + \int_{\partial e} \hat{T}\left[\left(CT^* - CT\right)n_t + \left(q^* - q\right)\cdot n_x\right] ds$$

$$+ \int_{\partial e} \hat{q}\left(\kappa T^* - \kappa T\right) n_x \, ds = 0$$

1F formulation for parabolic heat conduction equation

$$\hat{q} = -\kappa \nabla \hat{T}$$

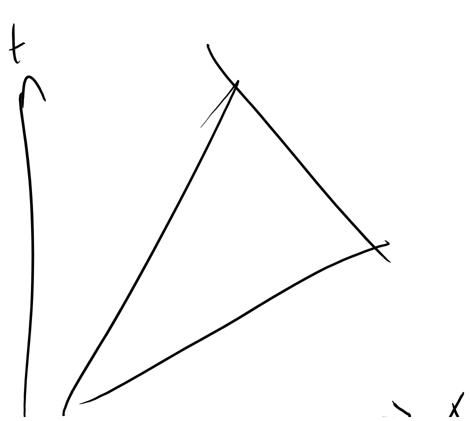
very similar to what we derived last time

(3) 2F hyperbolic eqn

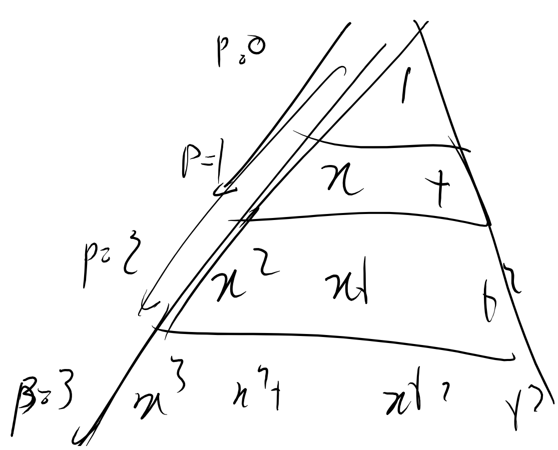
(4) 2F parabolic eqn

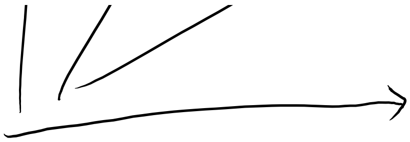
(5) 1F " eqn

How to solve it



T =





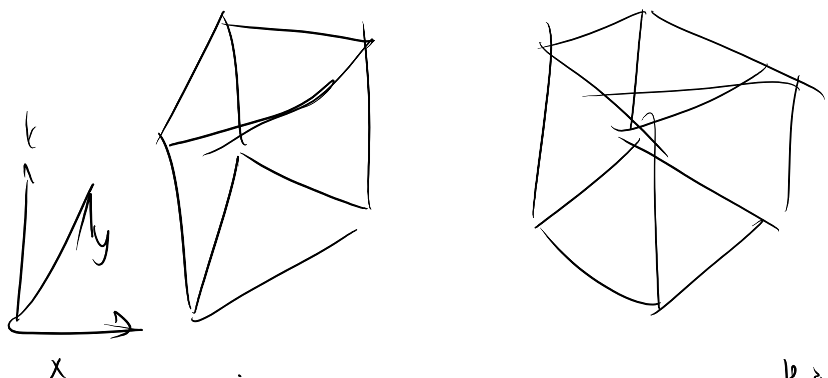
$\beta=3$ / $x^3 \quad x^2 t \quad x t^2 \quad t^3$

$$p=1 \qquad T = \underbrace{\overset{\text{constants}}{a_1} + a_2 x + a_3 t} \longrightarrow Ka = F$$

space DG

$$T = a_1(t) \cdot 1 + a_2(t)\, x \qquad p=1$$

$$C\dot{a} + Ka = F$$



p = 2 in space
p = 1 in time

$p_x = 2$
$p_t = 1$

$$\underbrace{\begin{matrix} & & 1 & & \\ & x & & y & \\ x^2 & & xy & & y^2 \end{matrix}} \times \begin{bmatrix} 1 \\ t \end{bmatrix}$$

12 terms

$$\begin{matrix} & & 1 & & \\ & x & & y & \\ x^2 & & xy & & y^2 \\ & x^2y & & xy^2 & \\ & & x^2y^2 & & \end{matrix} \times \begin{bmatrix} 1 \\ t \end{bmatrix}$$

$9 \times 2 = 18$

We want to solve (3) - hyperbolic 2F heat equation, with aSDG (asynchronous spacetime DG)

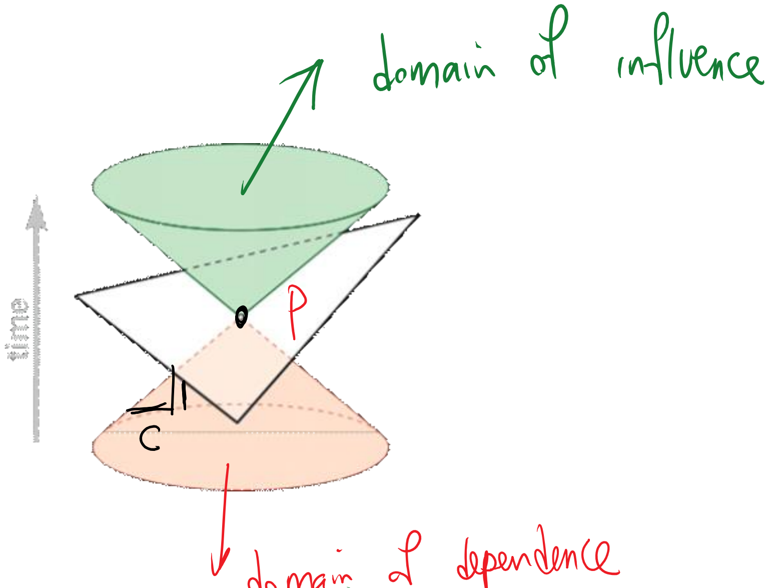
$$C\dot{T} + \nabla \cdot q = Q$$
$$\tau \dot{q} + k \nabla T = -q$$

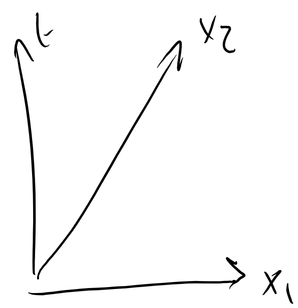
system of equation

wave speed $\boxed{c = \sqrt{\frac{k}{\tau C}}}$

$\tau \to 0$ $c \to \infty$ (as if we are reaching parabolic PDE limit where information moves with infinite speed!

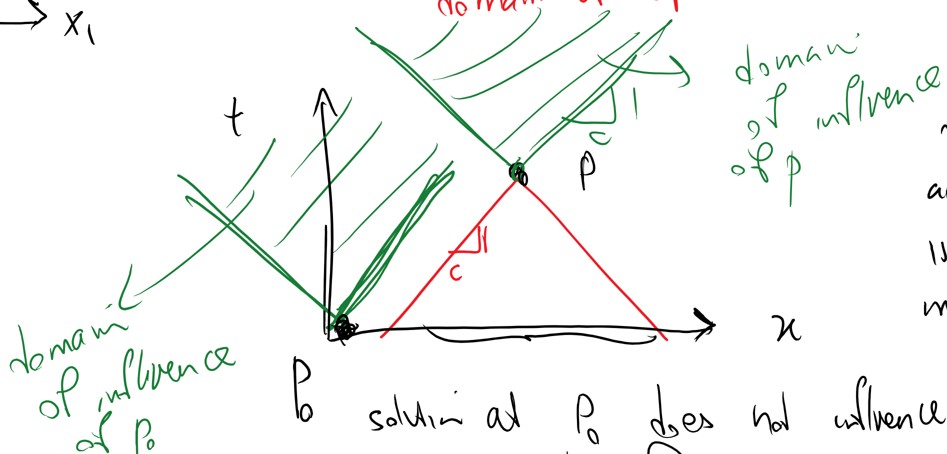
hyperbolic PDEs





2D

1D

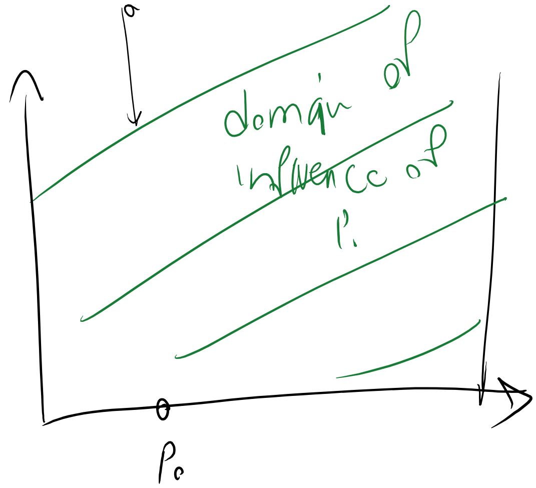


figures are for isotropic materials

solution at P_0 does not influence the solution @ P solution @ p

That's for a hyperbolic PDE such as MCV equation.
How about parabolic PDEs:

$$C\dot{T} - \nabla\cdot k\nabla T = Q$$



What is the problem with parabolic PDEs
Instant propagation of information.
Moving faster than speed of light!

MCV

$$\tau C \ddot{T} + C\dot{T} - \nabla\cdot k\nabla T = Q + \tau\dot{Q}$$

wave speed $c = \sqrt{\frac{k}{\tau C}}$

Important at small spacetime scales

time ξ τC = $\sqrt{\frac{k\tau}{C}}$

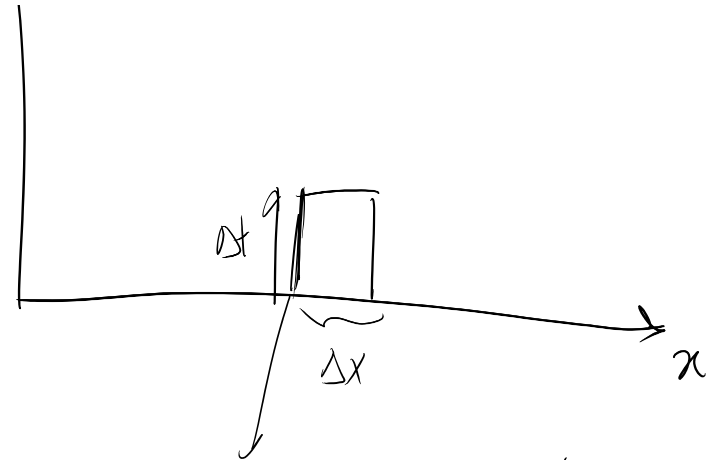
large space time scales

Meshing in ST

t

hyperbolic

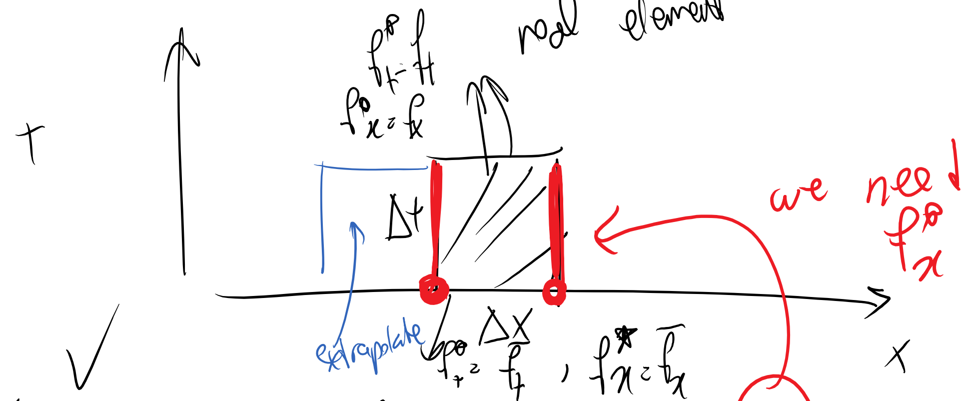
hyperbolic PDE



$c \qquad \Delta t < \frac{\Delta x}{c}$

conceptually is like time advance of an explicit DG for a hyperbolic problem with Block diagonal mass matrix

What if we want to formulate a <u>genuine</u> SDG



$$\int_e \hat{w}\left(f_t + \nabla \cdot f_x - S\right) dV + \int_{\partial e} w\left[\left(f_t^* - f_t\right) n_t + \left(f_x^* - f_x\right) n_x\right] = 0$$