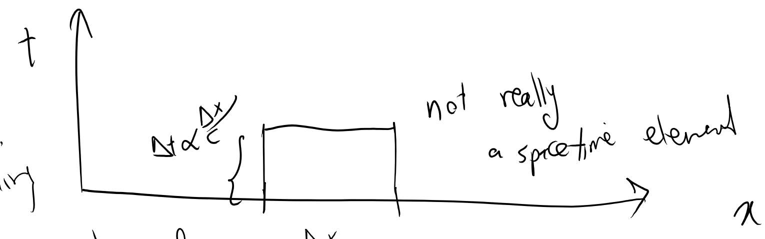
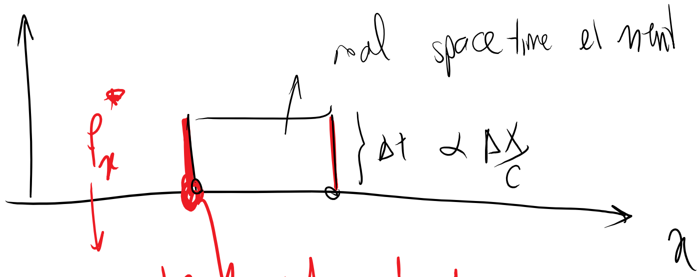
2018/03/21

Wednesday, March 21, 2018 11:41 AM

explicit time marching with block-diagonal mass matrix





can be value at t_n

or some more advanced projection of previous solution

we can also do the same process for a parabolic PDE

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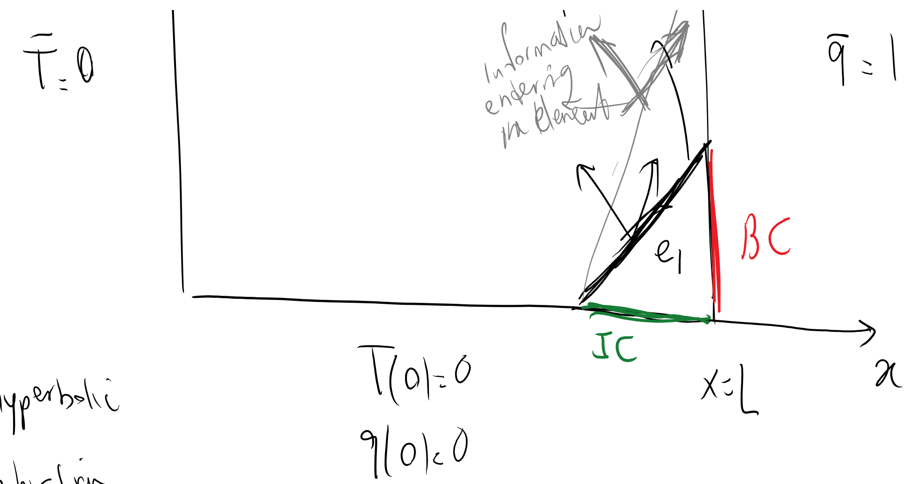
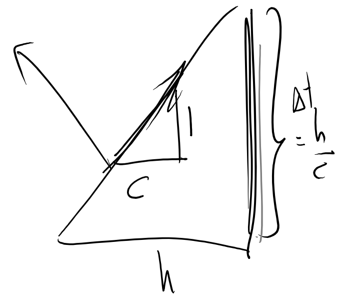
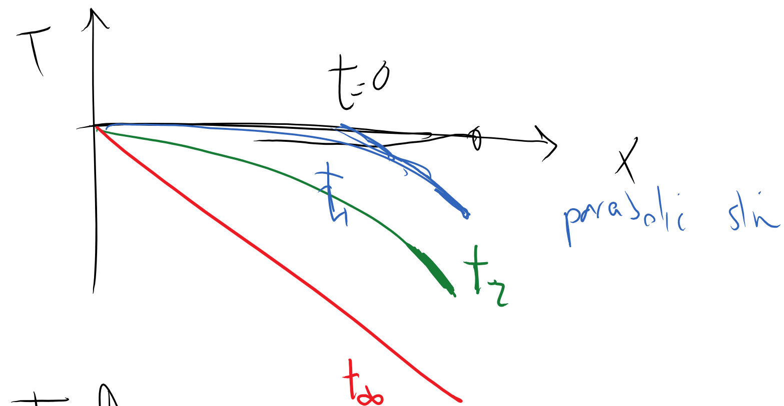
we'll focus on hyperbolic PDEs with the use of causal meshes

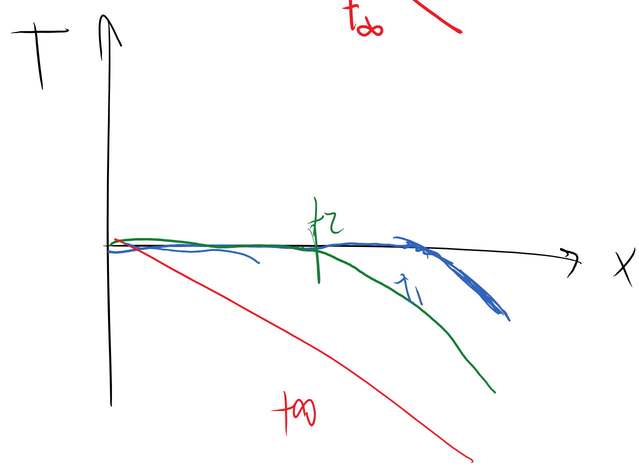


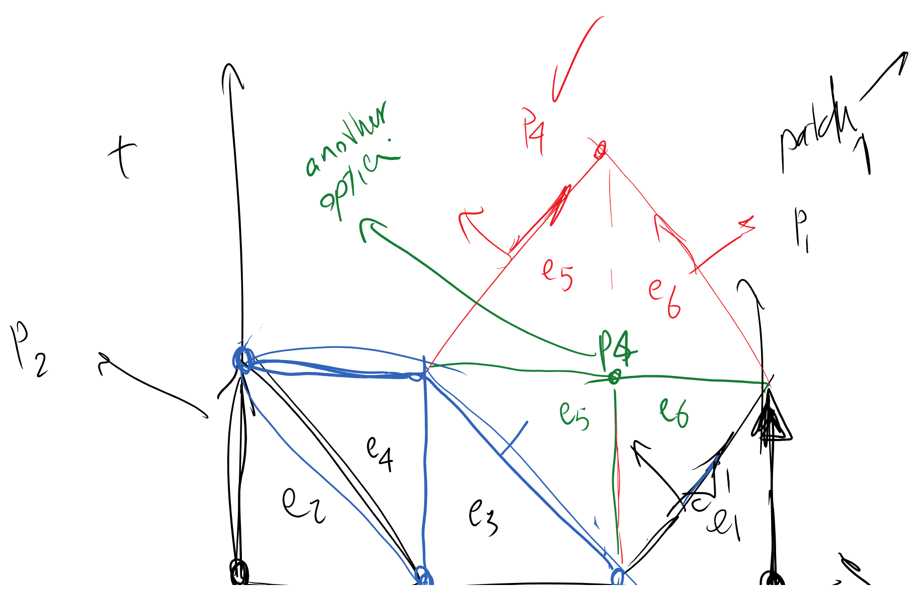


MCV hyperbolic heat conduction

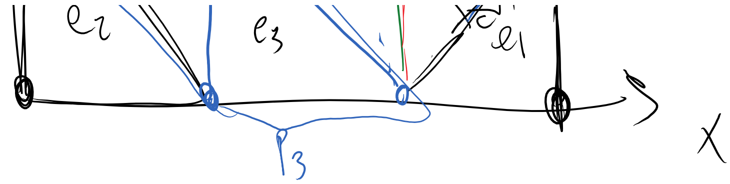


$$C = \sqrt{\frac{k}{C\tau}}$$

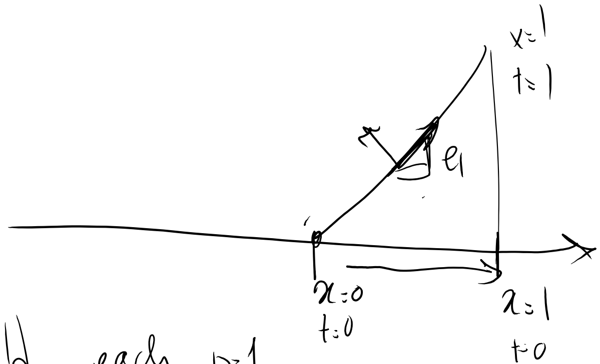


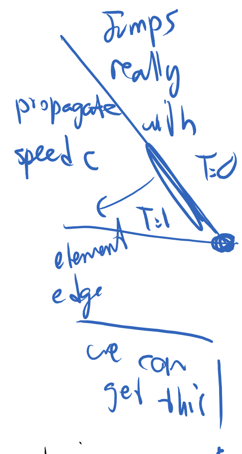


collection of elements that can be solved simultaneously



we are going to solve patch 1 (containing e_1) today:





MCV , 2 field , each $p=1$

$$T = a_1 + a_2 x + a_3 t$$

$$q = a_4 + a_5 x + a_6 t$$

①

Material properties

$$C = 1, \quad \kappa = 1, \quad \tau = 1$$

$$\rightarrow c = \sqrt{\frac{\kappa}{C\tau}} = 1$$

WRS:

interior integrals

$$\int_e \hat{T}\left(C\dot{T} + \nabla \cdot q - Q \right) dv + \int_{\partial e} \hat{T}\left[\left(CT^* - CT \right) n_t + \left(q^* - q \right) \cdot n_x \right] ds$$

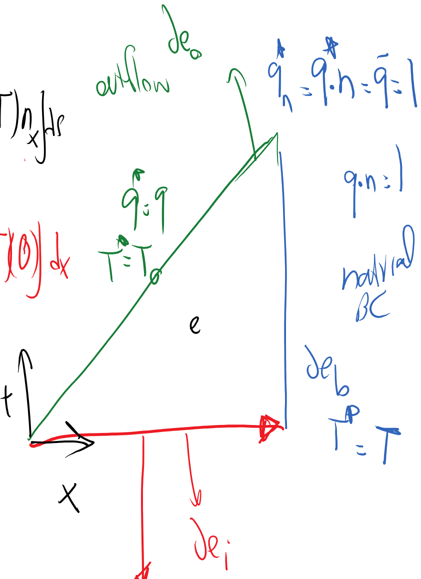
$$+ \int_e \hat{q}\left(\tau \dot{q} + \kappa \nabla T + q \right) dv + \int_{\partial e} \left[\hat{q}\left(\tau q^* - \tau q \right) n_t + \left(\kappa T^* - \kappa T \right) n_x \right] ds = 0$$

plugging $T = a_1 + a_2 x + a_3 t$ $\qquad C, k, \tau = 1$

$q = a_4 + a_5 x + a_6 t$

we do the boundary integrals first

$$\int_{\partial e} \left[\hat{T}(C T^* - CT) n_t + (q^* - q) n_x \right] ds + \int_{\partial e} \hat{q} \left(\underbrace{\tau q^* - \tau q}_{\text{doesn't exist on } \partial e_b} \right) n_t + (kT^* - kT) n_x \, ds$$



$$= \int_{\partial e_i} \hat{T}\big((1)(0) - (1)(T)\big)(-1) + \big((0) - q\big)(0) \, dx + \int_{\partial e_i} \hat{q}\big((1)(0) - (1)(q)\big)(-1) + (1)(0) - (1)T(0) \, dx$$

$\hat{q}$

$$+ \int_{\partial e_b} \left[\hat{T}\cancel{\big((1)T - (1)T\big)}_{0} + \big((1) - q \cdot (1)\big) \right] ds + \int_{\partial e_b} \hat{q}(0) + \big((1)T - (1)T\big)(1) \, ds$$

$n_t = 0$ on ∂e_b

$$+ \int_{\partial e_o} \hat{T}\left(\cancel{T - T} + \cancel{(q - q)}\right) n_x \, ds + \int \hat{q}\left(\cancel{(\tau q - \tau q)} n_t + (kT - kT) n_x\right) ds$$

$0 \qquad 0$

$n_x = 0$
$n_t = -1$

$T^* = T_o = 0$
$q^* = q_o = 0$

$\Rightarrow$ $\boxed{\text{contributions from the boundaries are in eqn (1)} \quad \int_{\partial e_i} (\hat{T} T + \hat{q} q) \, dx + \int_{\partial e_b} \hat{q}(1 - q) \, dt}$ (2a)

we need to add interior contributions $(Q = 0)$

(1) $\rightarrow$ $$\int_e \hat{T}(\dot{T} + \nabla \cdot q) \, dv + \int \hat{q}(\dot{q} + T_{,x} + q) \, dv$$

$T = a_1 + a_2 x + a_3 t \rightarrow \dot{T} = a_3$
$T_{,x} = a_2$

$q = a_4 + a_5 x + a_6 t \rightarrow \dot{q} = a_6$
$q_{,x} = a_5$

$\rightarrow$ $\int \hat{T}(\overset{\dot{T}}{\tilde{\ }} \ \overset{q_{,x}}{\widetilde{\ }} \ \ \hat{q}(\ \ \ a_5 x + a_6 t) \, dv$ 2b

$$\rightarrow \boxed{\int_e \hat{T}\,(\underbrace{a_3+a_5}_{T}) + \hat{q}\,(a_6+a_2+a_4+a_5 x+a_6 t)\,dV \qquad 2b}$$

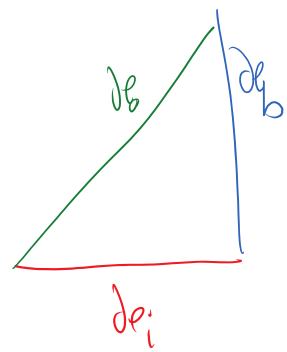
interior terms

Add 2a, 2b to get

$$\int_e \left\{ \hat{T}(a_3+a_5) + \hat{q}\,(a_6+a_2+a_4+a_5 x+a_6 t) \right\} dV$$

$$+ \int_{\partial e_i} \left[\hat{T}\cdot(\overbrace{a_1+a_2 x+a_3 t}^{T}) + \hat{q}\,(\overbrace{a_4+a_5 x+a_6 t}^{q}) \right] dx$$

$$\int_{\partial e_b} \hat{q}\,[1-(\overbrace{a_4+a_5 x+a_6 t}^{q})]\,dt = 0$$

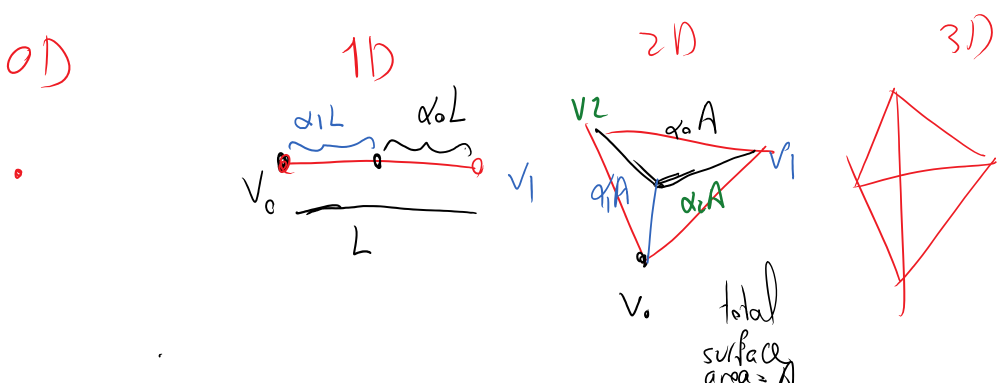


$$\hat{T} = \begin{bmatrix} 1 \\ x \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{q} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x \\ t \end{bmatrix}$$

6 eqns & 6 unknowns

Useful formulas for integration of constant, linear, and second order functions inside a triangle and on the boundaries -> useful for the HW assignment

Natural coordinates for simplices



d: dimension

surface area = A

d: dimension

$$\sum_{i=0}^{d} \alpha_i = 1 \qquad 1\ \alpha \text{ is redundant}$$

4

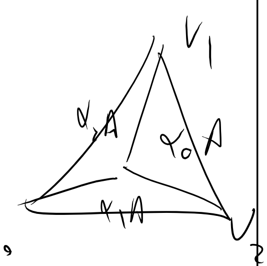
$$\boxed{\int_{T} \alpha_0^{p_0} \cdots \alpha_d^{p_d} = \frac{d!\, p_0! \cdots p_d!}{(d+p_0+p_1+\cdots+p_d)!}\, \underbrace{|M|}_{\text{Measure}}}$$

T: d dimensional simplex

Measure: length 1D, area 2D, volume 3D, ...

For a triangle

$$\int_{T} \alpha_0^{p_0} \alpha_1^{p_1} \alpha_2^{p_2}\, dV = |A| \frac{2!\, p_0!\, p_1!\, p_2!}{(2+p_0+p_1+p_2)!}$$



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Since our elements are $p=1$ → integrands are order 2 (maximum)

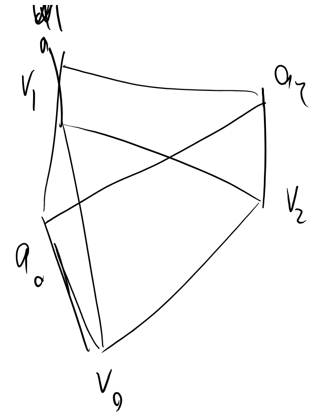
we want to find closed-form formulas for integration of 0, 1, 2 order functions over a triangle

~~a~~ constant

$$\int_{\text{triangle} \leftarrow A} c \; dA = c|A|$$

— linear $f(\alpha) = a_0\alpha_0 + a_1\alpha_1 + a_2\alpha_2$

$$f(@v_i) = a_i \quad \left(@v_i \quad \alpha_i = 1,\ \alpha_j = 0 \atop j \neq i\right)$$



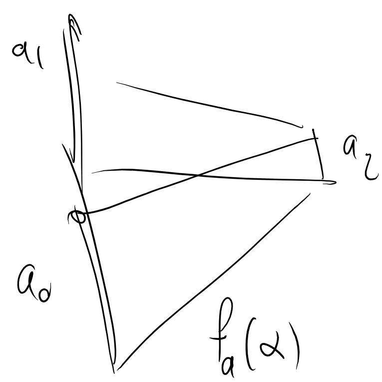
$$\int f(\alpha)\,dV = \int_A (a_0\alpha_0 + a_1\alpha_1 + a_2\alpha_2)\,dA = a_0\int_A \alpha_0\,dA + a_1\int\alpha_1\,dA + a_2\int\alpha_2\,dA$$

$$\int \alpha_0\,dA = \int \alpha_0^{\overset{p_0}{1}}\ \alpha_1^{\overset{p_1}{0}}\ \alpha_2^{\overset{p_2}{0}}\,dA = |A|\,\frac{2!\,p_0!\,p_1!\,p_2!}{(2+p_0+p_1+p_2)!} = \frac{2!\,1!}{3!}|A|$$

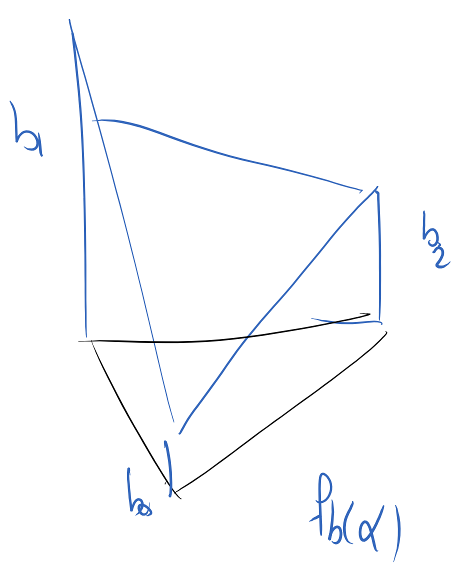
$\Rightarrow$ Same with $\int \alpha_1\,dA = \int \alpha_2\,dA = \frac{|A|}{3}$ $= \frac{|A|}{3}$

$$\int_A a_0\alpha_0 + a_1\alpha_1 + a_2\alpha_2 = \left(\frac{a_0 + a_1 + a_2}{3}\right)|A|$$

— linear $\times$ linear function



$\times$



$$\int f_a(\alpha)\, f_b(\alpha)\,dA = \int (a_0\alpha_0 + a_1\alpha_1 + a_2\alpha_2)(b_0\alpha_0 + b_1\alpha_1 + b_2\alpha_2)\,dA$$

$$\int_A f_a(\alpha)\, f_b(\alpha)\, dA = \int_A (a_0\alpha_0 + a_1\alpha_1 + a_2\alpha_2)(b_0\alpha_0 + b_1\alpha_1 + b_2\alpha_2)\, dA$$

$$= \int_A \Big[a_0 b_0\, \alpha_0^2 + a_1 b_1 \alpha_1^2 + a_2 b_2 \alpha_2^2 \Big] + \Big((a_0 b_1 + a_1 b_0)\alpha_0\alpha_1 + (a_0 b_2 + a_2 b_0)\alpha_0\alpha_2 + (a_1 b_2 + a_2 b_1)\alpha_1\alpha_2 \Big)$$

$$\int_A \alpha_0^2\, dA = \int \alpha_0^{\overset{P_0}{2}} \alpha_1^{\overset{P_1}{0}} \alpha_2^{\overset{P_2}{0}} = \frac{2!\,(\overset{P_0}{2}!)(\overset{P_1}{0}!)(\overset{P_2}{0}!)}{(2 + \underset{P_0}{2} + \underset{P_1}{0} + \underset{P_2}{0})!} = \frac{1}{6}$$

$$\int \alpha_0\alpha_1\, dA = \int \alpha_0^{\overset{P_0}{1}} \alpha_1^{\overset{P_1}{1}} \alpha_2^{\overset{P_2}{0}} = \frac{2!\,1!\,1!\,0!}{(2+1+1+0)!} = \frac{1}{12}$$

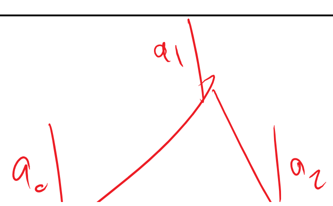
$$\rightarrow \int_A f_a(\alpha) f_b(\alpha)\, dA = \Big[\frac{1}{6}(a_0b_0 + a_1b_1 + a_2b_2) + \frac{1}{12}(a_0b_1 + a_1b_0 + a_1b_2 + a_2b_1 + a_2b_0 + a_0b_2) \Big] |A|$$

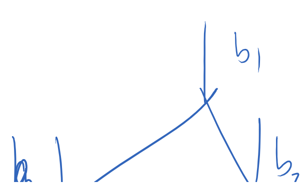
2nd order

1st order $\int_A f_a(\alpha)\, dA = \left(\frac{a_0 + a_1 + a_2}{3}\right)|A|$

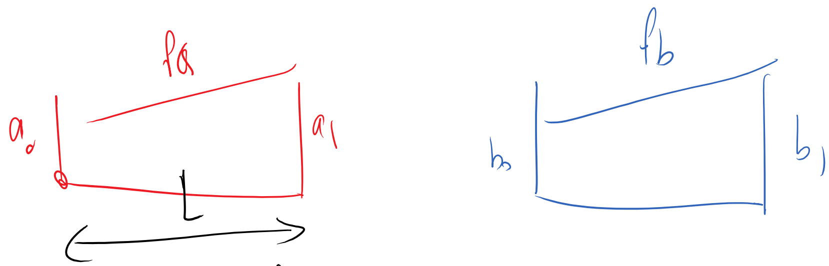
0th order $\int_A c\, dA = c\,|A|$

for triangle (6a)





1D (line) version of this



2nd order $\int f_a f_b \, ds = \left[\frac{a_0 b_0 + a_1 b_1}{3} + \frac{a_0 b_1 + a_1 b_0}{6} \right] L$

1st order $\int f_a \, ds = \left(\frac{a_0 + a_1}{2} \right) L$

0th order $\int c \, ds = c L$

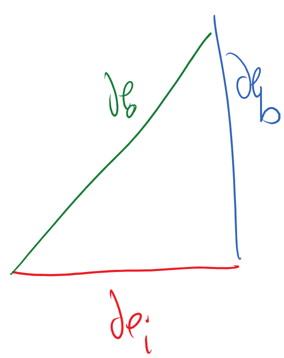
6b

Equation (*) copied:

$$\int_e \left\{ \hat{T}(a_3 + a_5) + \hat{q}\,(a_6 + a_2 + a_4 + a_5 x + a_6 t) \right\} dV$$

$$+ \int_{\partial e_i} \left[\hat{T} \cdot \Big(\overbrace{a_1 + a_2 x + a_3 t}^{T} \Big) + \hat{q} \Big(\overbrace{a_4 + a_5 x + a_6 t}^{q} \Big) \right] dx$$

$$\int_{\partial e_b} \hat{q} \Big[1 - \big(\overbrace{a_4 + a_5 x + a_6 t}^{q} \big) \Big] dt = 0$$



$$\hat{T} = \begin{bmatrix} 1 \\ x \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \hat{q} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x \\ t \end{bmatrix}$$

6 eqns & 6 unknowns

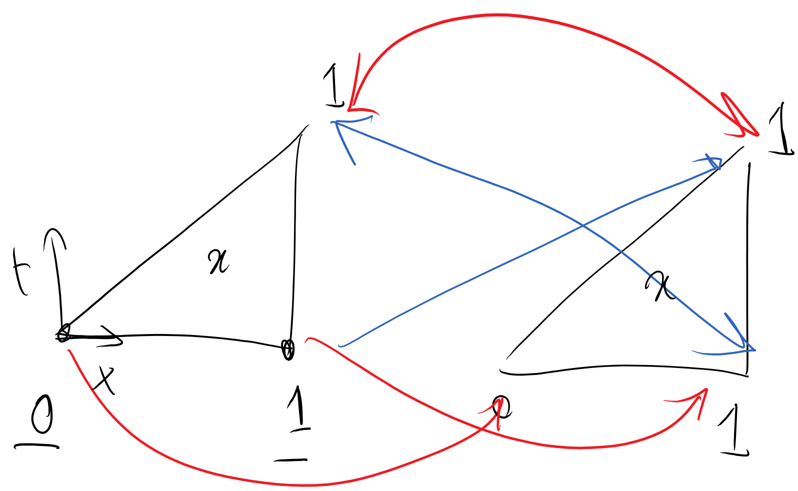
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ x \\ t \end{bmatrix}$$

2nd order ones $\qquad \int x \cdot x \, dv \qquad \int x . t \, dv$

$$\int t . t \, dv$$

1st order $\qquad \int x \, dv \qquad \int t . dv$

$$\int x . x \, dv = ?$$



$$\int_A x \cdot x \, dA = |A| \left(\frac{|x| + |x|}{6} + \frac{|x| + |x|}{12} \right) = \frac{A}{2}$$

$$\int x . t \, dA = |A| \left(\frac{|x|}{6} + \frac{|x|}{12} \right) = \frac{A}{4}$$