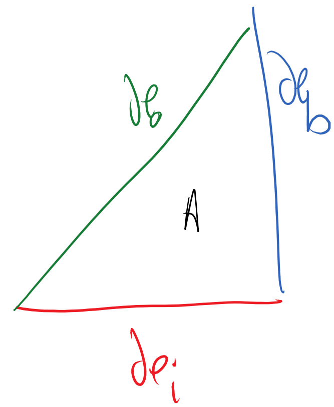
2018/03/26

Monday, March 26, 2018 11:38 AM

$$\int_e \left\{ \hat{T}(a_3+a_5) + \hat{q}\,(a_6+a_2+a_4+a_5 x+a_6 t) \right\} dV$$

$$+ \int_{\partial e_i} \left[\hat{T}\cdot(\overbrace{a_1+a_2 x+a_3 t}^{T}) + \hat{q}\,(\overbrace{a_4+a_5 x+a_6 t}^{q}) \right] dx$$

$$\int_{\partial e_b} \hat{q}\left[1 - (\overbrace{a_4+a_5 x+a_6 t}^{q}) \right] dt = 0$$



Other integrals:

$$\int_e x\,dA = \frac{2}{3}A \qquad \int t\,dA = \frac{1}{3}A$$

$$\int x^2\,dA = \frac{1}{2}A \qquad \int t^2\,dA = \frac{1}{6}A$$

$$\int xt\,dA = \frac{1}{4}A$$

$$I_i = \underbrace{\int\!\!\int}_{\hat{T}\,e} \left[\begin{bmatrix} 1 \\ x \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix} (a_3+a_5) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x \\ t \end{bmatrix} (a_6+a_2+a_4+a_5 x+a_6 t) \right] dA$$

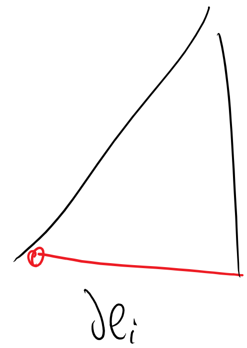
interior integral

$$I = A \left(\begin{bmatrix} a_3+a_5 \\ \frac{2}{3}(a_3+a_5) \\ \frac{1}{3}(a_3+a_5) \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ (a_6+a_2+a_4)+a_5 x^2/3+a_6(t/\ldots) \\ \vdots \end{bmatrix} \right)$$

$$
I_i = A\left(\begin{bmatrix} \tfrac{1}{3}(a_3+a_5) \\ \tfrac{1}{3}(a_3+a_5) \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} (a_6+a_2+a_4)+a_5 x\tfrac{2}{3} + a_6\left(\tfrac{1}{3}\right) \\ (a_6+a_2+a_4)\tfrac{2}{3} + a_5\tfrac{1}{2} + a_6\tfrac{1}{4} \\ (a_6+a_2+a_4)\tfrac{1}{3} + a_5\tfrac{1}{4} + a_6\tfrac{1}{6} \end{bmatrix}\right) \quad 1a
$$

Inflow face:

$$
I_{inf} = \int_{\partial e_i} \left[\hat{F}\left(a_1 + a_2 x + \cancel{a_3 t}\right) + \hat{q}\left(a_4 + a_5 x + \cancel{a_6 t}\right)\right] dx
$$

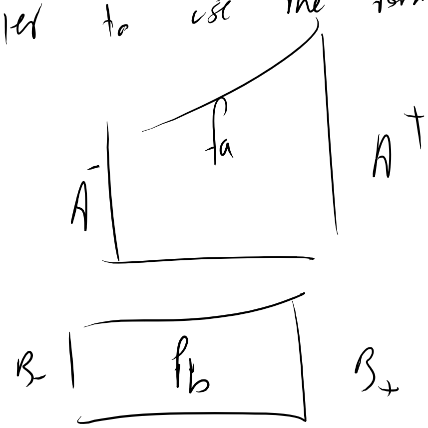


$t = 0$

for this simple case we directly integrate it in x but

for slant faces it's easier to use the formula

$$
\int_L f_a \times f_b \, ds = \frac{L}{6}\left(2A^-B^- + 2A^+B^+ + A^-B^+ + A^+B^-\right)
$$

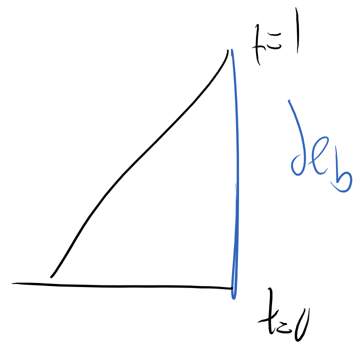


for this simple case

$$
I_{nf} = \int_0^1 \left(\begin{bmatrix} 1 \\ x \\ t \\ 0 \\ 0 \\ 0 \end{bmatrix}\left(a_1 + a_2 x + \cancel{a_3 t}\right) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x \\ t \end{bmatrix}\left(a_4 + a_5 x + \cancel{a_6 t}\right)\right) dx
$$

$$I_{int} = \begin{bmatrix} a_1 + a_2/2 \\ a_1/2 + a_2/3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 + a_5/2 \\ a_4/2 + a_5/3 \\ 0 \end{bmatrix} \quad (1b)$$

Finally we have the boundary integral on the boundary of the domain



$$I_b = \int_{\partial e_b} \hat{q}\left(1 - a_4 - a_5 x - a_6 t\right) dt$$

$x = 1$

$$= \int \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x \\ t \end{bmatrix} \left(1 - a_4 - a_5(1) - a_6(t)\right) dt$$

$$\rightarrow \quad \boxed{I_b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 - a_4 - a_5 - a_6/2 \\ 1 - a_4 - a_5 - a_6/2 \\ \ldots \end{bmatrix}} \quad (1c)$$

$$\partial b \quad \begin{pmatrix} 1 + a_4 & -a_5 & -a_6/2 \\ 1/2 & -a_4/2 & -a_5/2 & -a_6/3 \end{pmatrix}$$

these terms will go to the RHS

1c

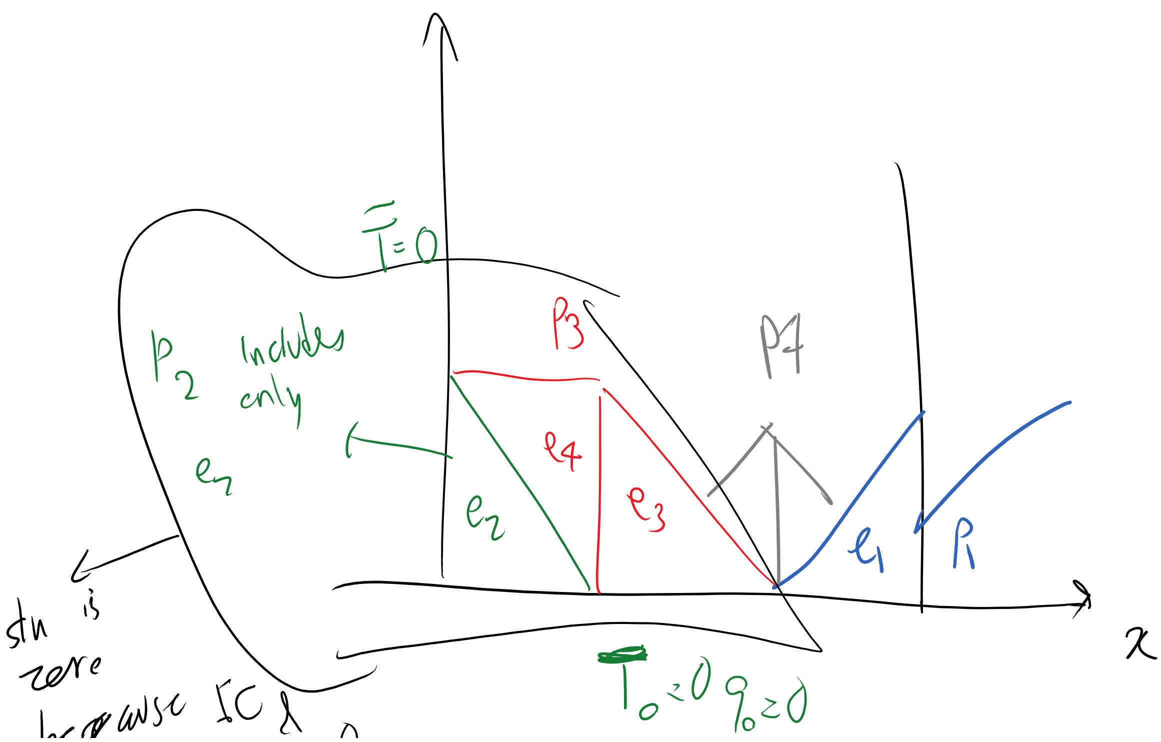
$$I = I_i + I_{\partial e_i} + I_{\partial b} = 0 \Rightarrow$$

$$K_{6\times 6}\, a_{6\times 1} = F_{6\times 1} \longrightarrow$$

$$\boxed{a = \begin{bmatrix} 0 \\ 0 \\ -3.27 \\ -6 \\ 3.27 \\ 15.27 \end{bmatrix}}$$

$$T = -3.27\,t$$

$$q = -6 + 3.27x + 15.27t$$



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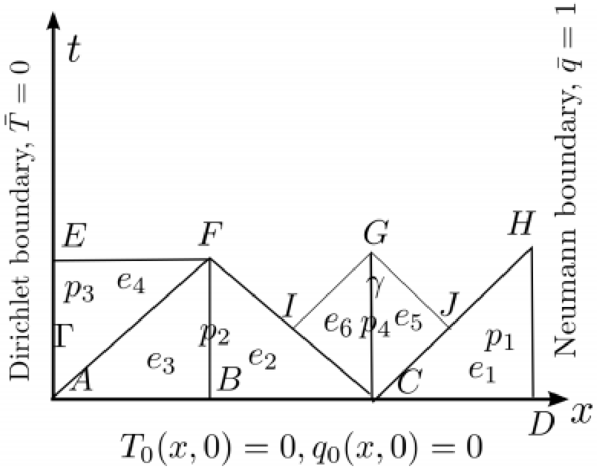


$$T^* = \frac{T_L + T_R}{2}$$

$$q^* = \frac{q_L + q_R}{2}$$

average flux

Riemann solutions:

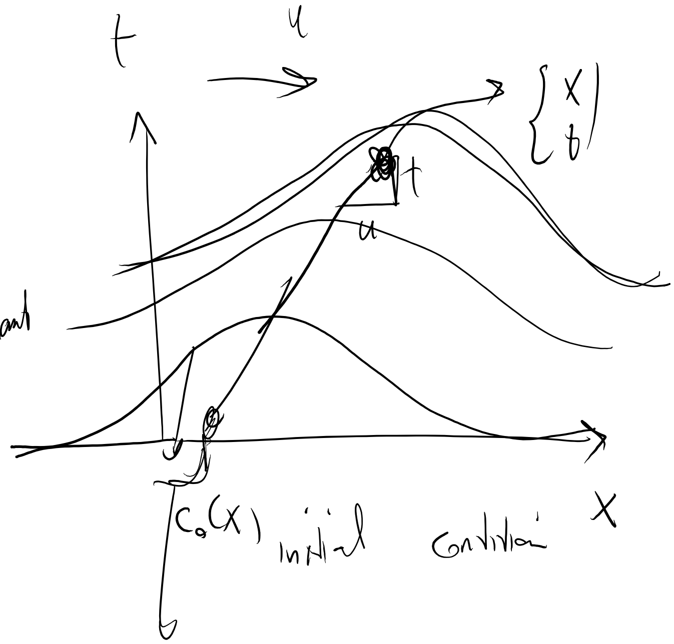
Let's solve the following advection problem:

$$\dot{c} + (cu)_{,x} = 0$$

c: density of something, e.g. pollutant

u: speed of propagation

assume u is constant

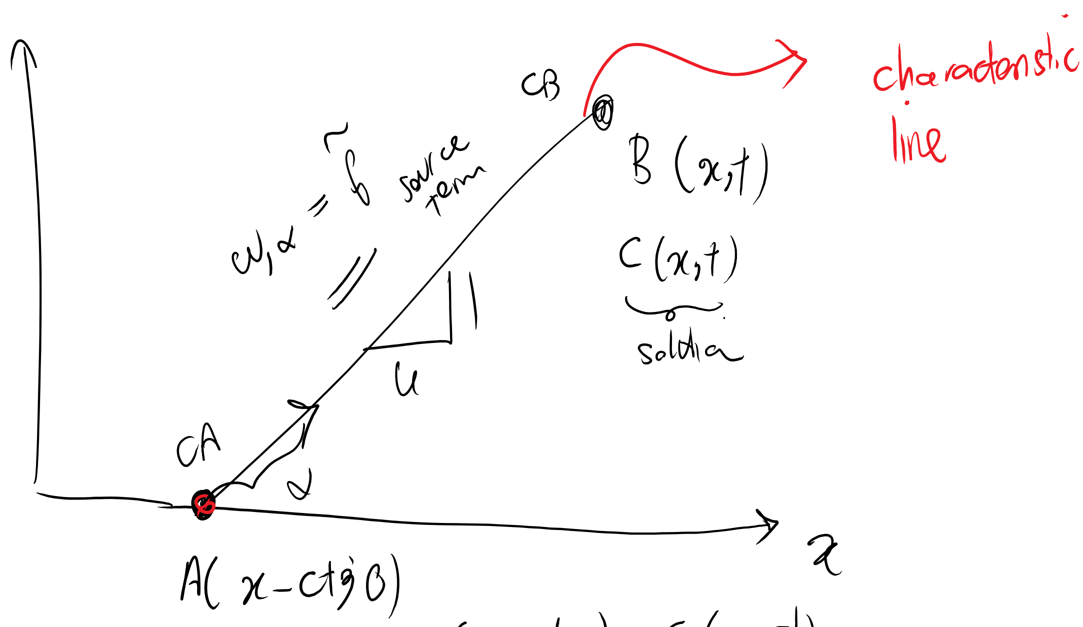


$$c(x,t) = c_0(x - ut)$$

$$\dot{c} + u\, c_{,x} = c_0' \frac{\partial\, x-ut}{\partial t} + u\, c_0' \frac{\partial\, x-ct}{\partial x} =$$

u: constant

$$c_0'(-u) + u(c_0') = 0$$



$c(x-ct, 0) = c_0(x-ct)$

$\boxed{c_A = c_B}$ characteristic value

$\dot{c} + (u\,c)_{,x} = f$ source term

Characteristic value remains constant or changes following an ODE along characteristics

The use of characteristics for hyperbolic PDEs:

Along characteristics PDE turns to an ODE which is easy to solve. For no source term case, characteristics in fact remain constant.

How about solid mechanics in 1D:

$$\dot{p} - \nabla \cdot \sigma = \rho b$$ equation of motion

$$\dot{p} - \nabla\cdot\sigma = \rho b$$ equation of motion

1D $$\boxed{\dot{p} - \sigma_{,x} = \rho b}$$ 1 eqn two unknowns p, σ

the next equation should be like

$$\dot{\sigma} + \bigcirc p_{,x} = ?$$

$$\dot{\varepsilon} = \overbrace{(u_{,x})}^{\circ} = (\dot{u})_{,x} = v_{,x} = \frac{1}{\rho}(\rho v)_{,x} = \frac{1}{\rho} p_{,x} \quad \rho \text{ constant}$$

$$\dot{\sigma} = \underbrace{E\dot{\varepsilon}} = \frac{E}{\rho} p_{,x}$$

system of conservation laws is

$$\boxed{\begin{aligned} \dot{p} - \sigma_{,x} &= \rho b \\ \dot{\sigma} - \frac{E}{\rho} p_{,x} &= 0 \end{aligned}} \quad (3)$$

nonzero for nonconstant properties

$$q = \begin{bmatrix} p \\ \sigma \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ -\frac{E}{\rho} & 0 \end{bmatrix} \qquad S = \begin{bmatrix} \rho b \\ 0 \end{bmatrix}$$

$$q = \begin{bmatrix} \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} & \\ -\frac{E}{\rho} & 0 \end{bmatrix} \quad S = \begin{bmatrix} \\ 0 \end{bmatrix}$$

primary unknowns

spatial flux matrix

source term

$$\dot{q} + A\, q_{,x} = S$$

Right way of writing a linear conservation law is

$$\dot{q} + (A_x q)_{,x} + (A_y q)_{,y} + (A_z q)_{,z} = S$$

$$\dot{q} + \underbrace{\nabla \cdot \left([A_x | A_y | A_z]\, q \right)}_{\text{complete divergence}} = S$$

for constant properties $\quad \dot{q} + A q_{,x} = S \quad \equiv \quad \dot{q} + (Aq)_{,x} = S$

How do we solve $\dot{q} + A q_{,x} = S$?

$$\begin{array}{l} \rho \\ \varepsilon \end{array} \dot{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}} + \underbrace{\begin{bmatrix} 0 & -1 \\ -E/\rho & 0 \end{bmatrix}}_{} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}_{,x} = \begin{bmatrix} \rho b \\ 0 \end{bmatrix}$$

non diagonal matrix

Let's hypothetically assume A is diagonal for $q_{n \times 1}$

$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} + \begin{bmatrix} A_{11} & & \\ & \ddots & \\ & & A_{nn} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}_{,x} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$$

$$\boxed{\dot{q}_1 + A_{11} q_{1,x} = s_1} \longrightarrow$$ what type of equation is this?

this is linear advection equation which we just solved for zero source term.

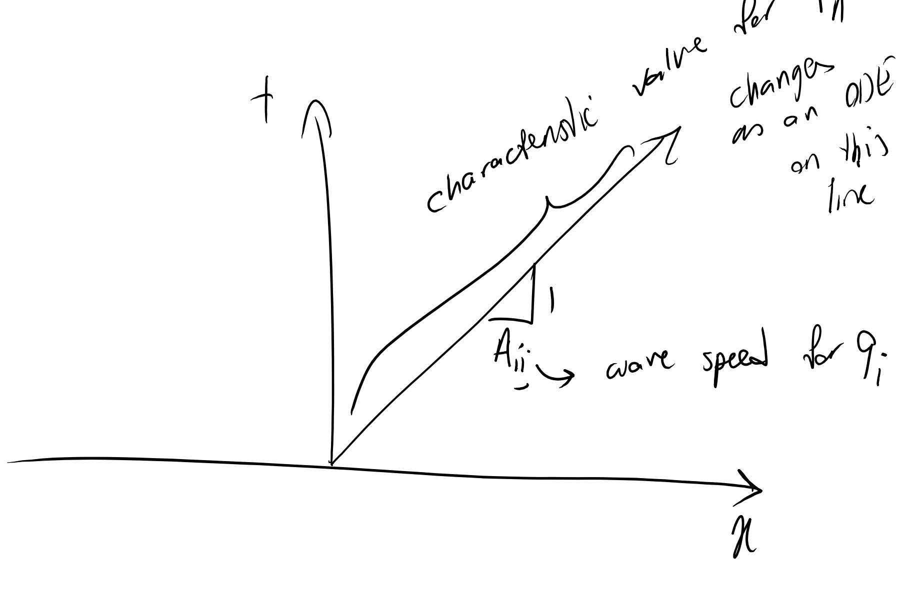
$\vdots$

$$q_n + A_{nn} q_{n,x} = s_n$$

$$q_i$$

$$q_i = q_{i0}(x - A_{ii} t)$$

for zero source term q_i



Similarly we get the solution for other q_i's.

q_i is a function of x, $\underline{A_{ii}}$, t

So if A is diagonal $q_t + Aq_{,x} = S$ is $\underline{\text{decoupled}}$

we can solve for each q_i independently.

What about

$$\boxed{q_t + Aq_{,x} = S} \quad (4)$$

when A is NOT diagonal?

the trick is eigenvalue decomposition

premultiply (4) by an $n \times n$ matrix U

$$U\dot{q} + UAq_{,x} = US$$

$$\overset{\circ}{(Uq)} + UA(U^{-1}U)q_{,x} = US$$

$$\underbrace{(Uq)}_{\displaystyle w} + UAU^{-1}Uq_{,x} = US$$

$$\dot{w} + \underbrace{(UAU^{-1})}_{\text{What if this is a diagonal matrix?}} w_{,x} = S_w$$

new source term

$S_w = US$

$$UAU^{-1} = D \Rightarrow \boxed{UA = DU} \quad (5)$$

diagonal matrix of eigen values

$$\begin{bmatrix} u_1 \\ \hline u_2 \\ \hline u_3 \\ \hline \vdots \\ \hline u_n \end{bmatrix} A = \begin{bmatrix} D_{11} & & \\ & \ddots & \\ & & D_{nn} \end{bmatrix} \begin{bmatrix} (u_1)_{1\times n} \\ \hline u_2 \\ \hline \vdots \\ \hline u_n \end{bmatrix}$$

$$(u_i)_{1\times n} A_{n\times n} = D_{\underline{ii}} \, u_i \quad \text{no summation on } i$$

$\downarrow$ $1\times n$

LEFT eigenvector # i of A

D_{ii} eigenvalue # i of A

Note of Right eigenpairs we have

$$A(v_i)_{n\times 1} = \underbrace{D_{ii}}_{\text{eigenvalue}} v_i \qquad \begin{bmatrix} v_i \\ \\ \\ \end{bmatrix}_{n\times 1}$$

$$A\left[v_1 \middle| \cdots \middle| v_n\right] = \left[v_1 \middle| \cdots \middle| v_n\right]\begin{bmatrix} D_{11} & & \\ & \ddots & \\ & & D_{nn} \end{bmatrix}$$

$AV = VD$ $\quad V = \left[v_1 \middle| \cdots \middle| v_n\right]$ Right eigenvalue-vector pairs

$UA = DU$ $\quad U = \begin{bmatrix} u_1 \\ \hline \vdots \\ \hline u_n \end{bmatrix}$ Left " "

$$A^t U^T = U^T D^T \rightarrow A^t\left[u_1 \middle| \cdots \middle| u_n\right] = \left[u_1 \middle| \cdots \middle| u_n\right]\begin{bmatrix} D_{11} & & \\ & \ddots & \\ & & D_{nn} \end{bmatrix} \quad 6$$

(U_i, D_{ii}) left eigenvector, eigenvalue of A $\Longleftrightarrow$

(U_i, D_{ii}) left eigenvector, eigenvalue of A $\Longleftrightarrow$

(U_i^T, D_{ii}) right " " " A^t

$$W_{n\times 1} = U_{n\times n}\, q_{n\times 1}$$

$\downarrow$

characteristic values

The solution process:

(1) $\dot{q} + A q_{,x} = S$ $\qquad$ $UA = \Lambda U$, define $w = Uq$

diagonal matrix of eigenvalues

matrix of left eigenvectors

$\longrightarrow \quad \dot{w} + \Lambda w_{,x} = S_w \qquad S_w = US$

(2) Solve the uncoupled simple ODES for $w_1 \cdots w_n$

$w_i(x,t)$ ✓

(3) $q = U^{-1} w \longrightarrow q(x,t)$ will be known

(6)

We do this solution process through an example:

1D elastodynamics eqn was

$$\dot{q} + A q_{,x} = S$$

$$q = \begin{bmatrix} p \\ \epsilon \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ -\frac{E}{\rho} & 0 \end{bmatrix} \qquad S = \begin{bmatrix} \rho b \\ 0 \end{bmatrix}$$

$$q = \begin{bmatrix} \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} & \\ -E/\rho & 0 \end{bmatrix} \qquad \begin{bmatrix} \\ 0 \end{bmatrix}$$

Let's assume $ph = 0 \rightarrow S = 0$

— We need to solve eigenvalue & eigenvector pairs of A.

$$\begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -\frac{E}{\rho} & 0 \end{bmatrix} = \lambda \begin{bmatrix} u^1 & u^2 \end{bmatrix}$$

$$-E/\rho \, u^2 = \lambda u^1$$

$$-u^1 = \lambda u^2 \qquad \frac{E}{\rho} u^1 u^2 = \lambda^2 u^1 u^2 \rightarrow \lambda^2 = \frac{E}{\rho} \rightarrow \lambda = \underbrace{\pm\sqrt{\frac{E}{\rho}}}_{\text{wave speed in D}}$$

$u^1 \neq 0$, $u^2 \neq 0$ | otherwise $u^1 = u^2 = 0$

Eigenvectors

$$\lambda = -c = -\sqrt{\frac{E}{\rho}}$$

$$\begin{bmatrix} u^1 & u^2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -c^2 & 0 \end{bmatrix} = -c \begin{bmatrix} u^1 & u^2 \end{bmatrix}$$

$$-c^2 \downarrow E/\rho$$

$-c^2 u^2 = -c u^1$ $\qquad$ $u^2 = 1$

$= u^1 = -c u^2$ $\qquad$ $u^1 = c$ $\qquad$ will work

$$\left(\lambda = -c, \quad U = [c \quad 1] \right)$$

Similarly for $\lambda = c$

$$\left(\lambda = c, \quad U = [-c \quad 1] \right)$$

So

$$U = \begin{bmatrix} U_1 \\ \hline U_2 \end{bmatrix} = \begin{bmatrix} c & 1 \\ -c & 1 \end{bmatrix} \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix}$$