2018/03/28

Wednesday, March 28, 2018 11:36 AM

1D elastodynamics eqn was

$$\dot{q} + A q_{,x} = f$$

$$q = \begin{bmatrix} p \\ \sigma \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ -E/\rho & 0 \end{bmatrix} \qquad f = \begin{bmatrix} \rho b \\ 0 \end{bmatrix}$$

Let's assume $\rho b = 0 \rightarrow f = 0$

$$u A = \lambda u \quad \rightarrow \quad u (A - \lambda I) = 0$$

eigenvalues

$$\det(A - \lambda I) = 0 \quad \rightarrow \quad \det \begin{bmatrix} -\lambda & -1 \\ -\frac{E}{\rho} & -\lambda \end{bmatrix} = 0$$

same condition for right eigenvalues

$\rightarrow$ Left & Right eigenvalues are the same

$$\rightarrow \lambda^2 - \frac{E}{\rho} = 0 \quad \rightarrow \quad (1) \ \boxed{\lambda = \pm c \qquad c = \sqrt{\frac{E}{\rho}}}$$

Eigenvectors

$$\lambda_1 = -c \qquad [u_1 \ \ u_2] \begin{bmatrix} -(-c) & -1 \\ -c^2 & -(-c) \end{bmatrix} = [0 \ \ 0]$$

$\rightarrow \quad c u_1 - c^2 u_2 = 0$

$-u_1 + c u_2 = 0$

first eigenvector $\underline{u_1^{(1)} = c \quad u_2^{(1)} = 1}$

$\lambda_2 = c \longrightarrow$ 2nd eigenvector $u_1^{(2)} = -c \quad u_2^{(2)} = 1$

1st eigenvector

$$U = \begin{bmatrix} u^{(1)} \\ \hline u^{(2)} \end{bmatrix} \qquad D = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix}$$

2nd eigenvector

$$\boxed{U = \begin{bmatrix} c & 1 \\ -c & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix}}$$

$\dot{w} + D w_{,x} = 0 \rightarrow$ zero source term

$\dot{w}_1 - c\, w_{1,x} = 0$

$\dot{w}_2 + c\, w_{2,x} = 0$

$w = Uq$

characteristic values

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Initial Value problem

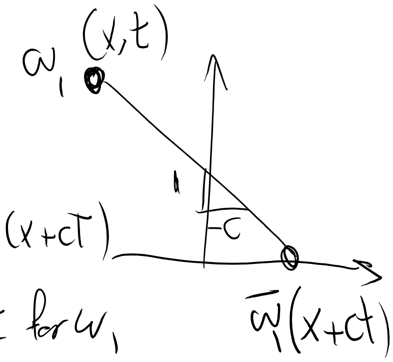
ⓐ $t=0$

$$\boxed{\begin{aligned} p(x,0) &= q_1(x,0) = \bar{q}_1(x) \\ \sigma(x,0) &= q_2(x,0) = \bar{q}_2(x) \end{aligned}}$$

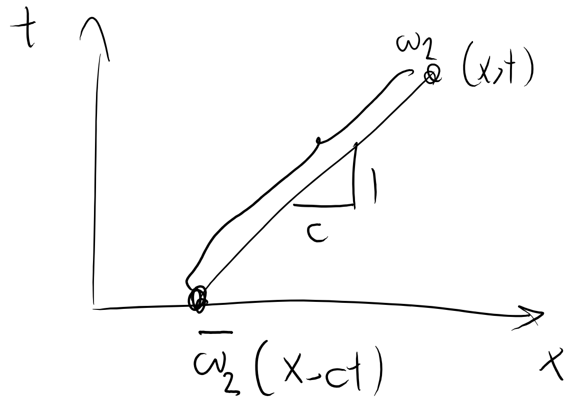
②

$$\dot{w}_1 - c\,w_{1,x} = 0 \longrightarrow w_1(x,t) = \bar{w}_1(x+ct)$$

IC for w_1



$$\dot{w}_2 + c\,w_{2,x} = 0 \longrightarrow w_2(x,t) = \bar{w}_2(x-ct)$$



$$\boxed{\begin{aligned} w_1(x,t) &= \bar{w}_1(x+ct) \\ w_2(x,t) &= \bar{w}_2(x-ct) \\ &\bar{w}_1, \bar{w}_2 \text{ are IC for } w_1, w_2 \end{aligned}}$$

③

What are $\bar{w}_1$, $\bar{w}_2$

$$w = Uq \longrightarrow w(x,0) = \bar{w}(x) = U q(x,0) = U\bar{q}(x)$$

$$\bar{w}(x) = \begin{bmatrix} \bar{w}_1(x) \\ \bar{w}_2(x) \end{bmatrix} = \begin{bmatrix} c_d & 1 \\ -c_d & 1 \end{bmatrix} \begin{bmatrix} \bar{q}_1(x) \\ \bar{q}_2(x) \end{bmatrix}$$

$$\left[\bar{\omega}_2(x)\right] \qquad \left[-c_d \quad 1\right]\left[\bar{q}_2(x)\right]$$

$$\boxed{\begin{aligned}\bar{\omega}_1(x) &= c_d\,\bar{q}_1(x) + \bar{q}_2(x)\\ \bar{\omega}_2(x) &= -c_d\,\bar{q}_1(x) + \bar{q}_2(x)\end{aligned}} \quad (4)$$

$$(3),(4) \Rightarrow \quad \left.\begin{aligned}\omega_1(x,t) &= \bar{\omega}_1(x+ct)\\ \omega_2(x,t) &= \bar{\omega}_2(x-ct)\end{aligned}\right\} \rightarrow$$

$$\boxed{\begin{aligned}\omega_1(x,t) &= c_d\,\bar{q}_1(x+ct) + \bar{q}_2(x+ct)\\ \omega_2(x,t) &= -c_d\,\bar{q}_1(x-ct) + \bar{q}_2(x-ct)\end{aligned}} \quad (5)$$

$$\Rightarrow \quad \begin{aligned}&q_1(\rho)\\ &q_2(\sigma)\end{aligned}$$

$$\left.\begin{aligned}\omega_1(x,t)\\ \omega_2(x,t)\end{aligned}\right\} \longrightarrow \quad \begin{aligned}q_1(x,t)\\ q_2(x,t)\end{aligned} \qquad \omega = Uq \rightarrow$$

$$q = U^{-1}\omega$$

$$U = \begin{bmatrix} c & 1 \\ -c & 1 \end{bmatrix} \rightarrow \quad U^{-1} = \frac{1}{2c}\begin{bmatrix} 1 & -1 \\ c & c \end{bmatrix}$$

$$2c \left[c \quad c \right]$$

$$U^{-1} = \begin{bmatrix} \frac{1}{2c} & -\frac{1}{2c} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2c} & -\frac{1}{2c} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \rightarrow$$

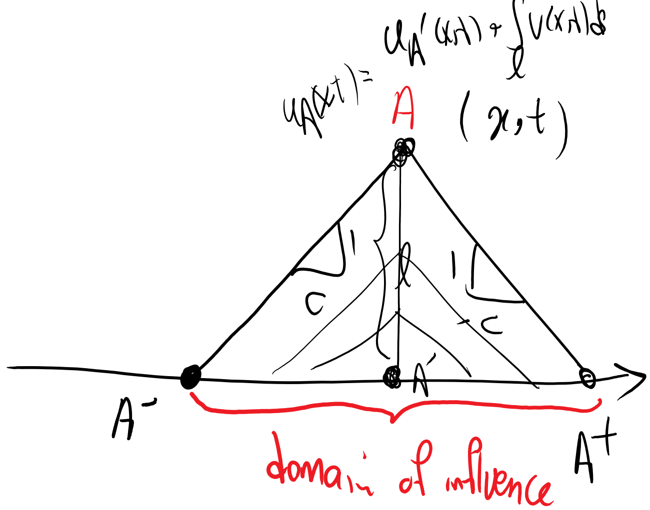
$$\boxed{\begin{aligned} q_1 &= \frac{1}{2c}\left(w_1 - w_2\right) \\ q_2 &= \frac{1}{2}\left(w_1 + w_2\right) \end{aligned}} \quad (6)$$

$$\boxed{\begin{aligned} w_1(x,t) &= c_d\, \bar{q}_1(x+ct) + \bar{q}_2(x+ct) \\ w_2(x,t) &= -c_d\, \bar{q}_1(x-ct) + \bar{q}_2(x-ct) \end{aligned}} \quad (5)$$

$$\rightarrow$$

$$q_1(x,t) = \frac{1}{2}\left(\bar{q}_1(\overbrace{x+ct}^{A^+}) + \bar{q}_1(\overbrace{x-ct}^{A^-})\right) + \frac{1}{2c}\left(\bar{q}_2(\overbrace{x+ct}^{A^+}) - \bar{q}_2(\overbrace{x-ct}^{A^-})\right)$$

$$q_2(x,t) = \frac{cd}{2}\left(q_1(\underbrace{x+ct}_{A^+}) - \bar{q}_1(\underbrace{x-ct}_{A^-})\right) + \frac{1}{2}\left(\bar{q}_2(\underbrace{x+ct}_{A^+}) + \bar{q}_2(\underbrace{x-ct}_{A^-})\right)$$



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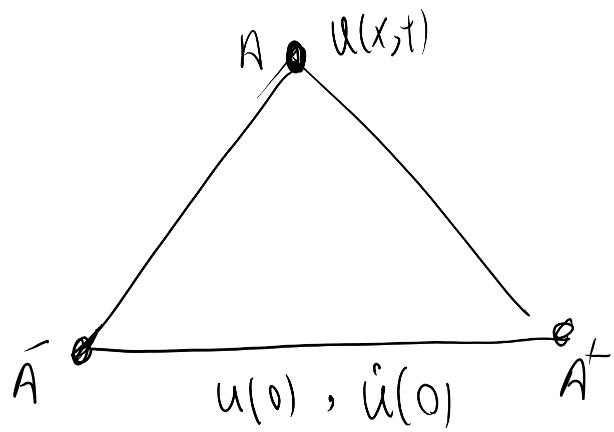
1. Solution @ A only depends on the solutions @ A^- & A^+

- what about displacement at point A

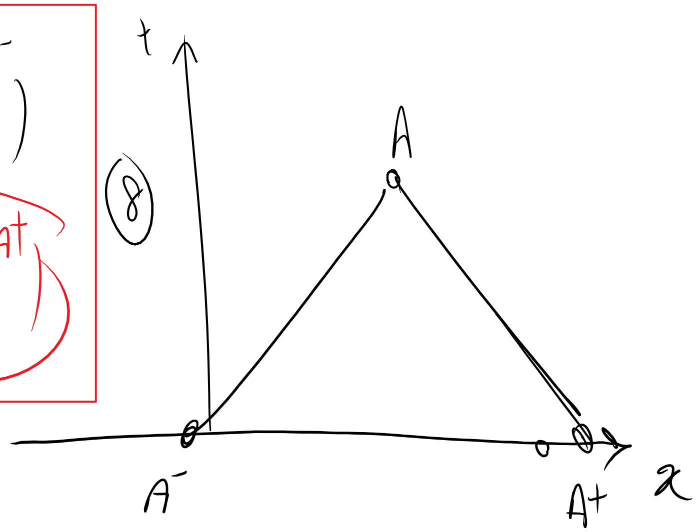


For the primary field u solution depends on all values between (and including) A-, A+, you can derive the equation for u yourself.

2. q1 and q2 are averages of their corresponding values

plus some jump terms of the other quantities:

$$
\boxed{
\begin{aligned}
q_1^A &= \frac{1}{2}\left(q_1^{A^-} + q_1^{A^+}\right) + \frac{1}{2c_d}\left(q_2^{A^+} - q_2^{A^-}\right) \\
q_2^A &= \frac{c_d}{2}\left(q_1^{A^+} - q_1^{A^-}\right) + \frac{1}{2}\left(q_2^{A^-} + q_2^{A^+}\right)
\end{aligned}
}
\qquad (8)
$$

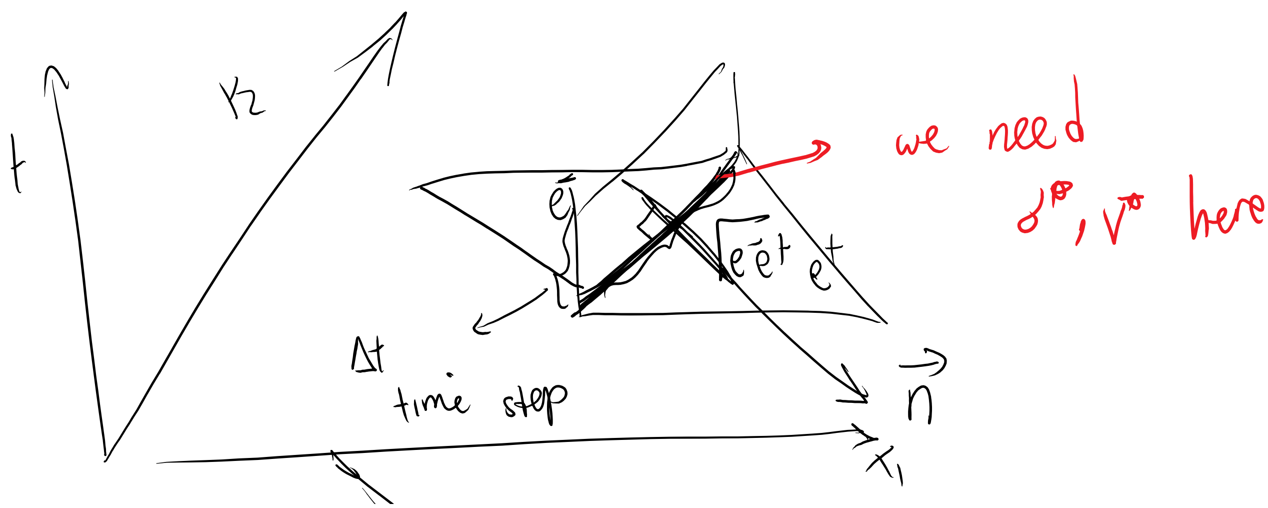


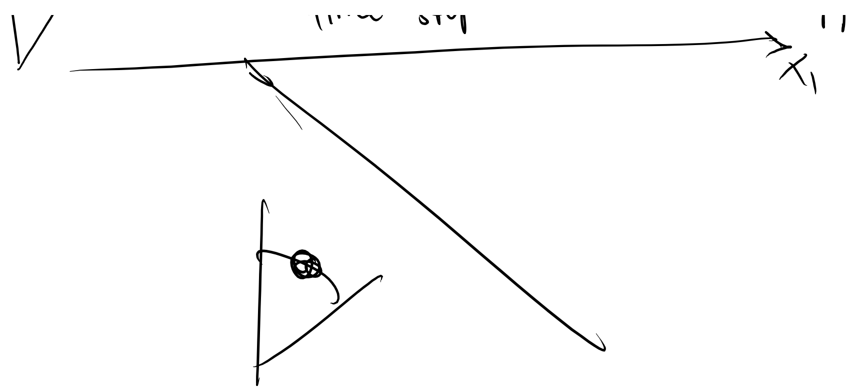
Numerical flux:
Recall the weak statement for local DG formulation of elastodynamic problem:

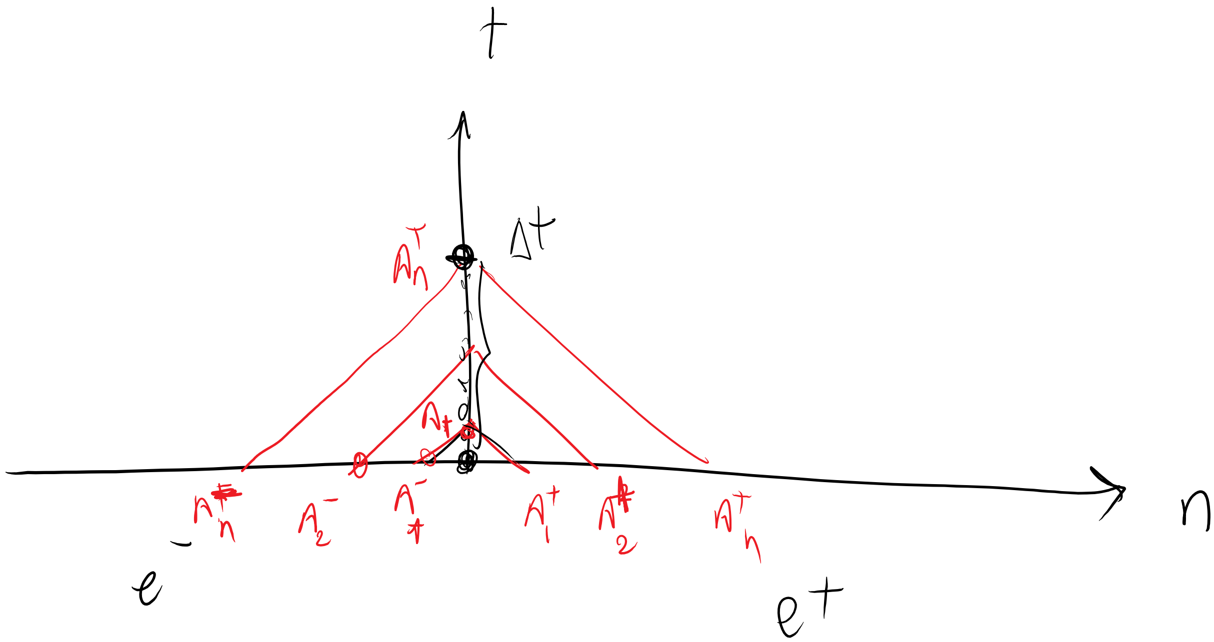
damping coef.

$$\int_e \hat{u}\left(\ddot{u} + d\dot{u} - \nabla\cdot\sigma - \rho b\right) dv \; +$$

$$\int_{\partial e} \hat{u}\left(-\sigma^* + \sigma\right)\cdot n \, ds + \lambda \int_{\partial e} \hat{q}\cdot n\left(-v^* + v\right) ds = 0$$





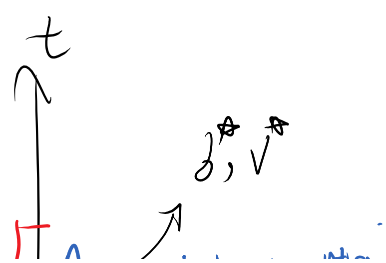


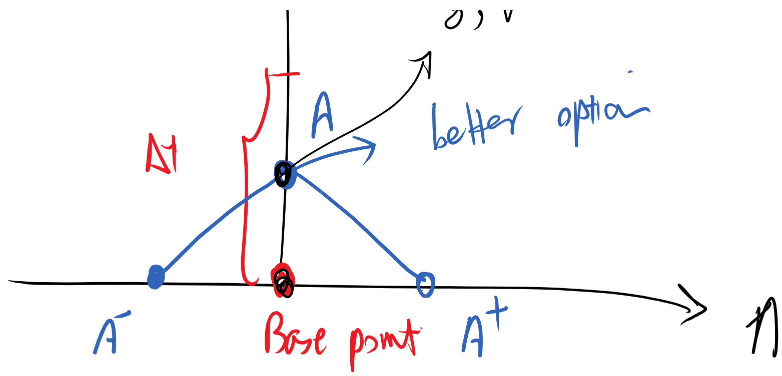
we need σ^*, v^*

Basically we need integration of fluxes from 0 to Δt

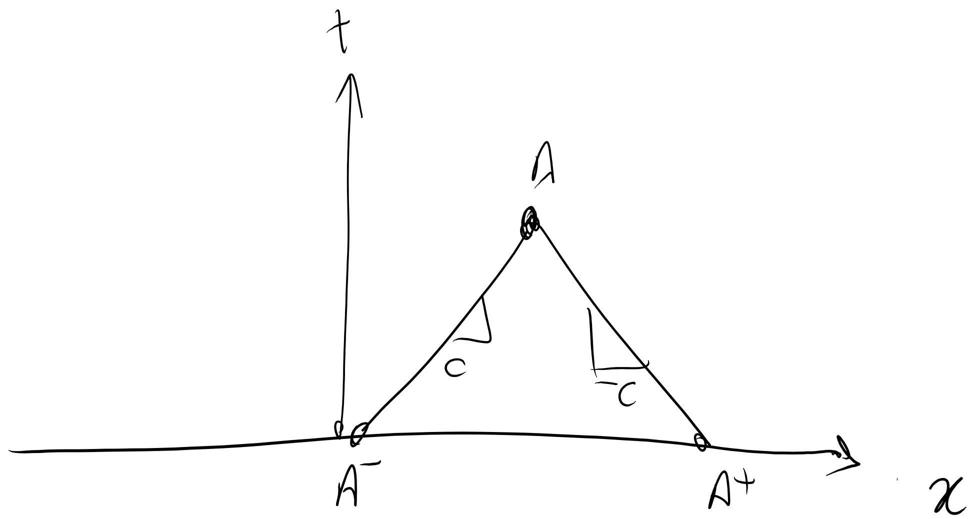
What if we want to use something simpler, that is computing the target (star) values at only 1 point.

What would you choose:





If we choose the mid-point in Delta t (that is tn + Delta t / 2), as in figure, the target values are computed from points A- and A+.



p^*

$$P_A = \frac{1}{2}\left(P_{A^-} + P_{A^+}\right) + \frac{1}{2c}\left(\sigma_{A^+} - \sigma_{A^-}\right)$$

σ^*

$$\sigma_A = \frac{c}{2}\left(P_{A^+} - P_{A^-}\right) + \frac{1}{2}\left(\sigma_{A^-} + \sigma_{A^+}\right)$$

are needed σ^*, v^*

how about v^* $\qquad p = \rho v$

$$V^{*} = V_A = \frac{p_A}{\rho} = \frac{1}{2}\left(V_A^{-} + V_A^{+}\right) + \frac{1}{2c\rho}\left(\sigma_A^{+} - \sigma_A^{-}\right)$$

$$\sigma^{*} = \sigma_A = \frac{c\rho}{2}\left(V_A^{+} - V_A^{-}\right) + \frac{1}{2}\left(\sigma_A^{-} + \sigma_A^{+}\right)$$

$$\boxed{\begin{aligned} &Z = c\rho \quad \text{impedance} \\ &V^{*} = \frac{1}{2}\left(V_A^{-} + V_A^{+}\right) + \frac{1}{2Z}\left(\sigma_A^{-} + \sigma_A^{+}\right) \\ &\sigma^{*} = \frac{Z}{2}\left(V_A^{+} - V_A^{-}\right) + \frac{1}{2}\left(\sigma_A^{-} + \sigma_A^{+}\right) \end{aligned}}$$

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Solutions for NO SOURCE TERM problem

point 3 : What are the characteristic values?

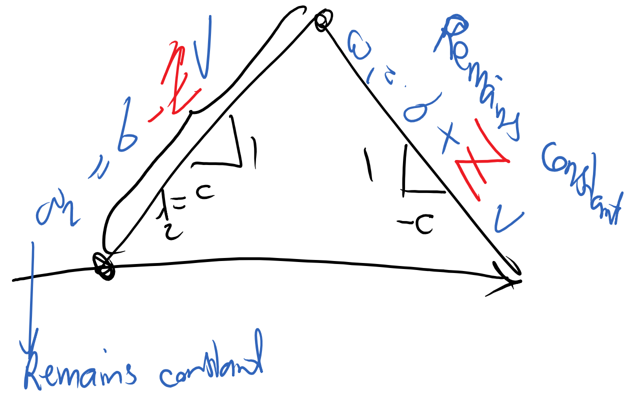
$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = U \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} c & 1 \\ -c & 1 \end{bmatrix} \begin{array}{c} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \end{array} \begin{array}{l} \to p \\ \to \sigma \end{array}$$

$$\omega_1 = cP + \sigma \qquad\qquad P = \rho V$$

$$\omega_2 = -cp + \sigma$$

$$\omega_1 = \sigma + Z v \longleftrightarrow \lambda_1 = -c$$

$$\omega_2 = \sigma - Z v \qquad \lambda_2 = c$$



Side note

2D, 3D $\quad \omega_1^i = t^i + Z^{(i)} v^i$ $\qquad$ no summation

$$\omega_2^i = t^i - Z^{(i)} v^i$$

$$\omega_1^{(1)} = t^1 + Z^{(1)} v^1$$

$$\omega_2^{(1)} = t^1 - Z^{(1)} v^1$$

$i = 1$ $\quad$ dilatational mode $\quad Z^{(1)} = c_d \rho$

λs are $= \mp c_d = \pm\sqrt{\frac{\lambda + 2\mu}{\rho}}$ $\quad$ dilational wave speed

$i = 2, 3$ $\quad$ shear modes $\quad Z^{(2)} = Z^{(3)} = c_s \rho$

λs are $= \pm c_s = \sqrt{\frac{\mu}{\rho}}$ $\quad$ shear wave speed

This holds for isotropic materials

Anisotropic (2D, 3D) $\quad \omega^i = t^i - \sum_j Z^{ij} v^j$

Anisotropic$(u_s, s_{ij}))\ \omega = c - \sum_j \Big(\sum \Big)\ V$

↓ matrix

2D $\qquad \sum_{2\times 2}$ matrix

3D $\qquad \sum_{3\times 3}$ matrix

] normal & shear get coupled

4) The effect of source terms:

We solve a simple example to show the effect of the source term.

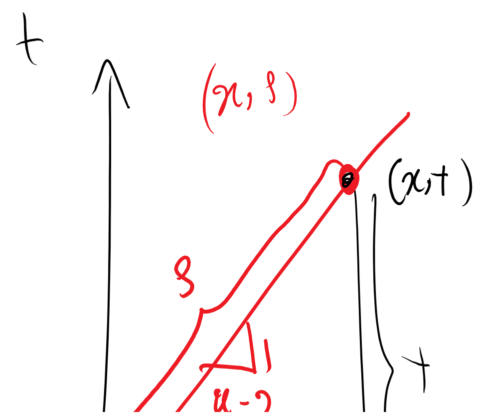
$$\dot{c} - u\, c_{,x} = f(x,t)$$

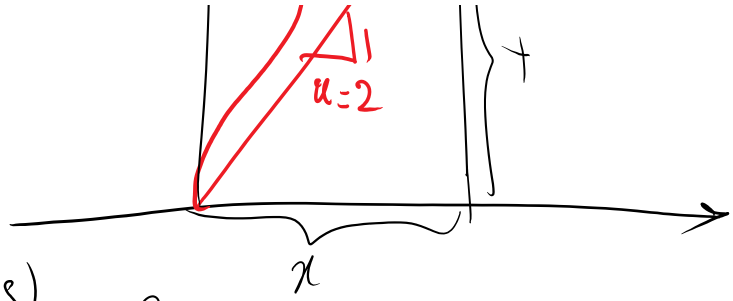
Simple 1D linear advection eqn

$u = 2$

$f(x,t) = x - t$

$c(x, t=0) = c_0(x)$





$$\dot{c} - 2c_{,x} = \frac{\partial c(x,s)}{\partial s} = f(x,t) = x - t$$

$$s = x - 2t \rightarrow t = \frac{x-s}{2} \rightarrow f(x,t) = x - \left(\frac{x-s}{2}\right) = \left(\frac{x+s}{2}\right)$$

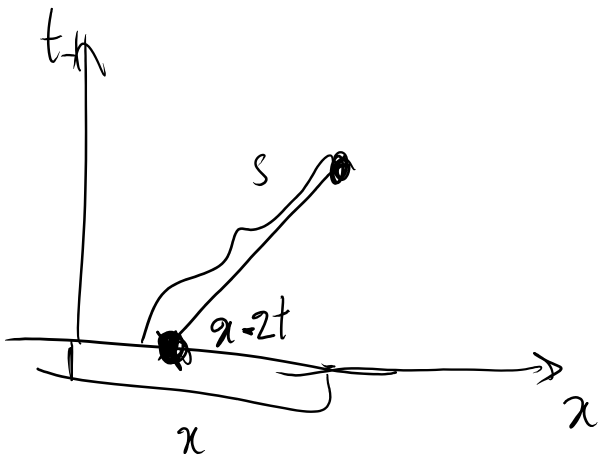
$$\frac{\partial c(x,s)}{\partial s} = \tilde{f}(x,s) = \left(\frac{x+s}{2}\right)$$

at $t=0$

$x - 2t = s \rightarrow$

$x - 2\times 0 = s$

@ $t=0$ $\quad \underline{x=s}$



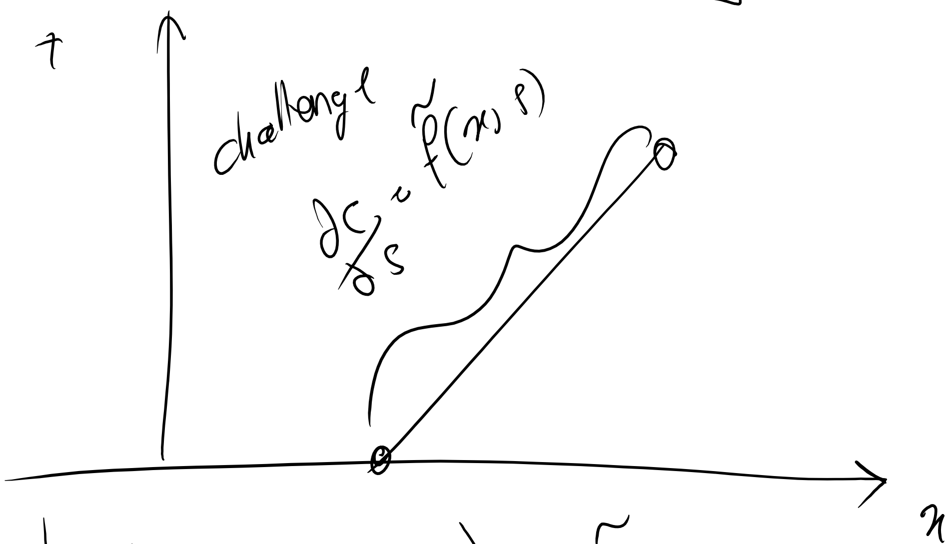
$$\frac{\partial c(x,s)}{\partial s} = \frac{x+s}{2} \rightarrow \boxed{c(x,s) = \frac{xs}{2} + \frac{s^2}{4} + g(x)}$$

$s=0 \quad c_0(x) = c(x,0) = g(x)$

@ IC $\quad x = s$

$$c(x,s) = \frac{xs}{2} + \frac{s^2}{4} + c_0(x)$$

$$\boxed{c(x,t) = \frac{x(x-2t)}{2} + \frac{(x-2t)^2}{4} + c_0(x)}$$



we needed to integrate $\frac{\partial c}{\partial s} = \tilde{f}(x,s)$

along the characteristic to get the value of solution (c)

What is the implication if say we have source term

In the electrodynamic problem

$$\dot{p} - b_{,x} = \rho b$$

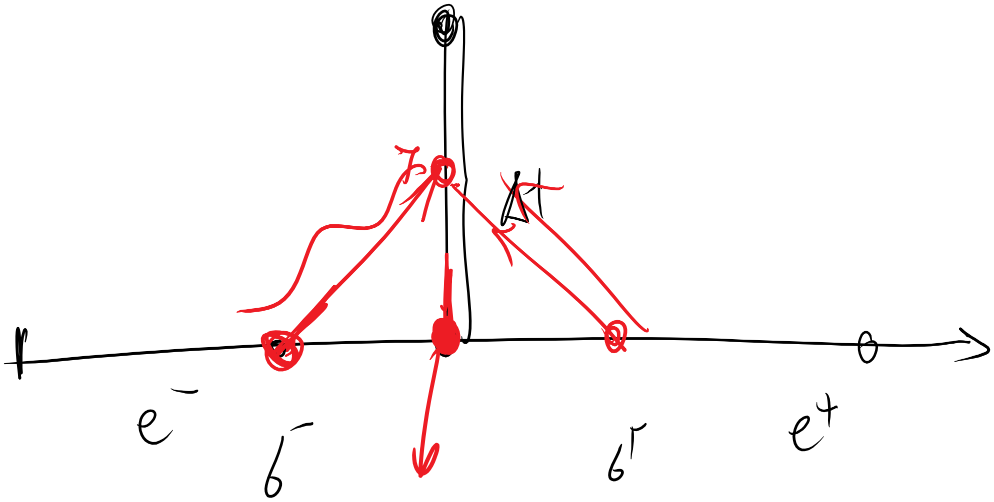
$$\dot{b} - \frac{E}{\rho} p_{,x} = 0$$

$$S = \begin{bmatrix} \rho b \\ 0 \end{bmatrix}$$

we

$\sigma - \frac{E}{\rho}\rho_{,x} = 0$

are assumed $\rho\sigma$ to be zero



simple

a age + jump term slns are had