2018/04/02

Monday, April 02, 2018 11:41 AM

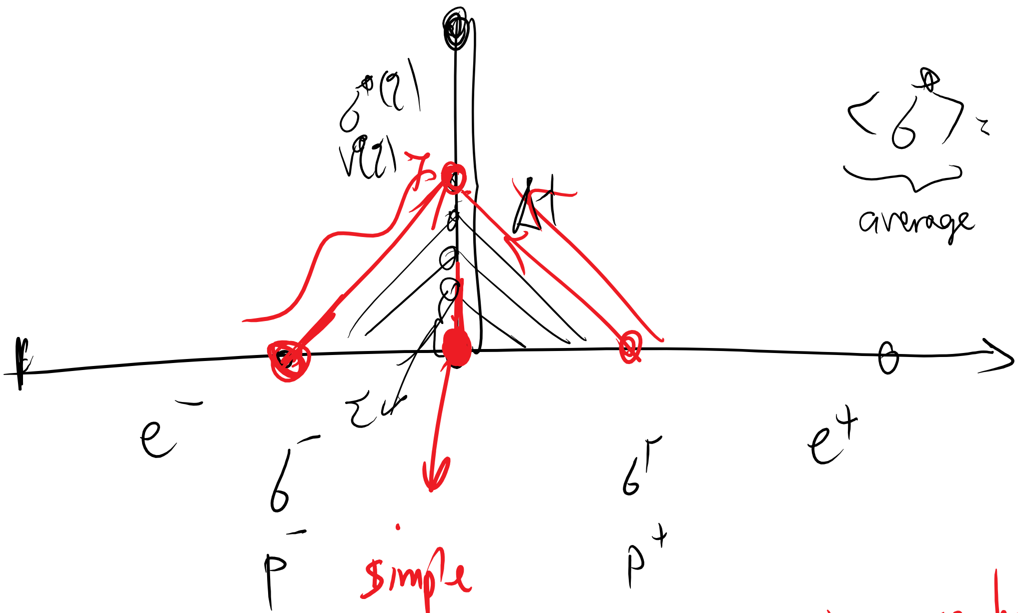
The difficulty in handling source terms when obtaining numerical fluxes (for example Riemann solution)

$$\dot{p} - b_{,x} = \rho b$$

$$\dot{b} - \frac{E}{\rho} p_{,x} = 0$$

$$S = \begin{bmatrix} \rho b \\ 0 \end{bmatrix}$$

we assumed ρb to be zero



a avg + jump term slns are hard

$$\underbrace{\langle \sigma^* \rangle}_{\text{average}} = \frac{1}{\Delta t} \int_0^{\Delta t} \sigma^*(\tau)\, d\tau$$

best

next best option

$$\langle \sigma^* \rangle \approx \sigma^*\left(\tau = \frac{\Delta t}{2}\right) +$$

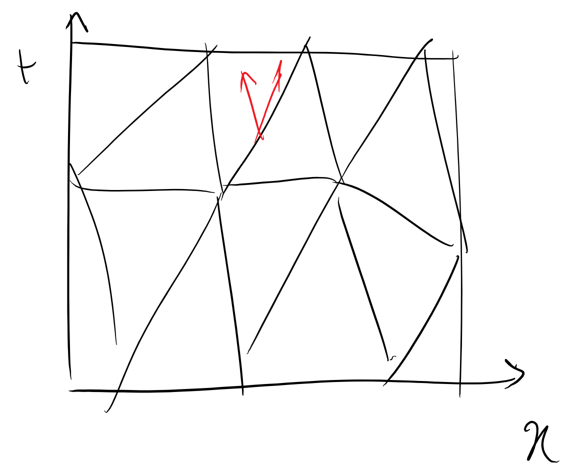
$$\langle \sigma^n \rangle \approx \sigma^n\left(\tau = \frac{\Delta t}{2}\right) + O(\Delta t^2)$$

In either case, the Riemann solution at $\tau \neq 0$

requires the integration of source terms along characteristics.

What happens to Riemann solutions for spacetime DG methods and in particular aSDG method.

① Spacetime DG methods:
FEM



we solve a globally coupled problem in $\mathbb{E}^d \times \mathbb{R}^+$

space dim → d

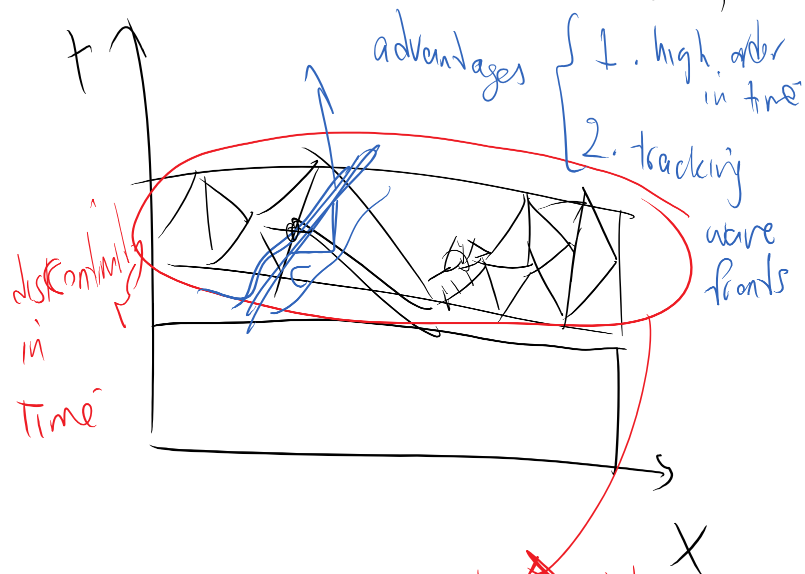
$\mathbb{R}^+$ → time

- The form of shape functions in spacetime for CFEM makes the whole domain globally coupled even if all

Next word

② TDG

Time discontinuous Galerkin method (Tom Hughes & Hulbert)



discontinuity between time slabs

enables solution of 1 time

makes the whole domain globally coupled even if all faces are causal.

-> DG formulation

Can employ causality of the mesh to solve one of a few elements at a time.

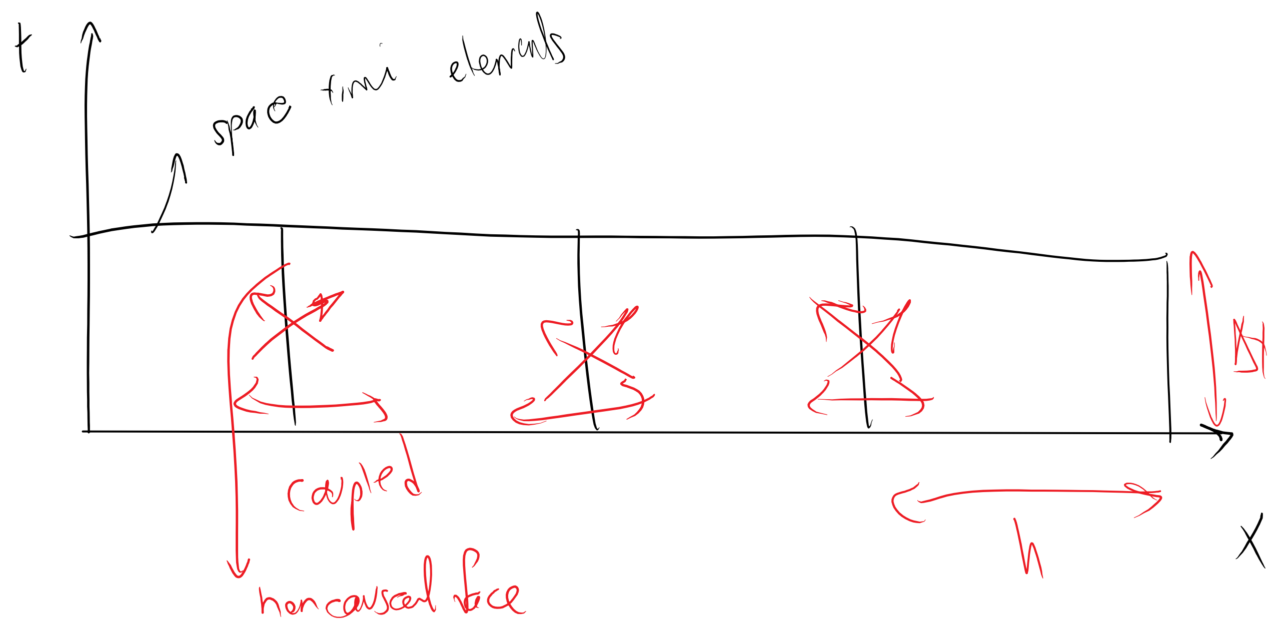
enables solution of 1 time slab rather than whole domain at a time

First SPACETIME methods used CFEM and solved a globally coupled problem.

- Clearly no star value for CFEMs.

③ Space time DG method   Implicit

Van der Vegt etc



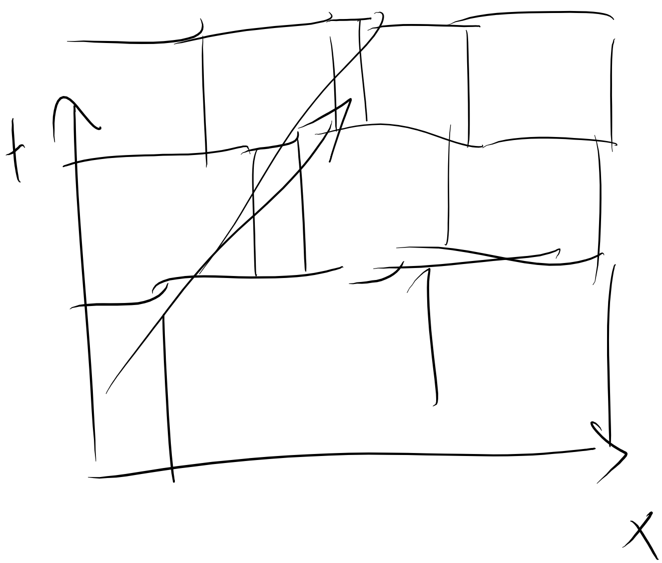
the whole domain is coupled even if a hyperbolic problem

$\underbrace{\Delta t_{max}}$ h for hyperbolic PDE

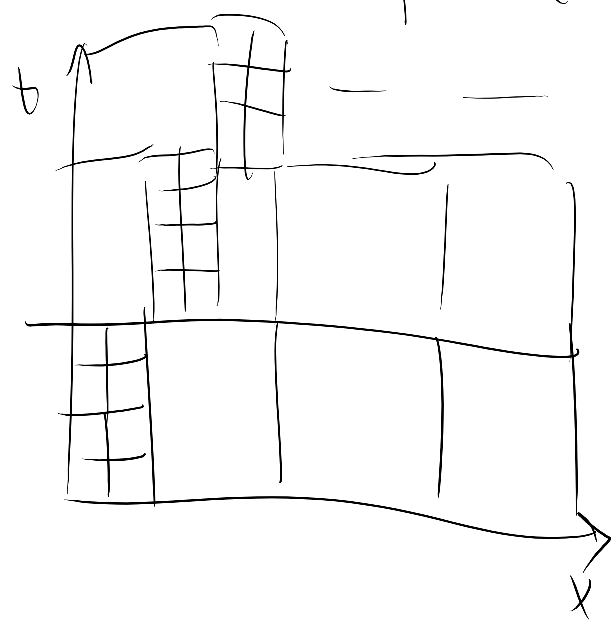
no time step limit

More refined versions of SDG

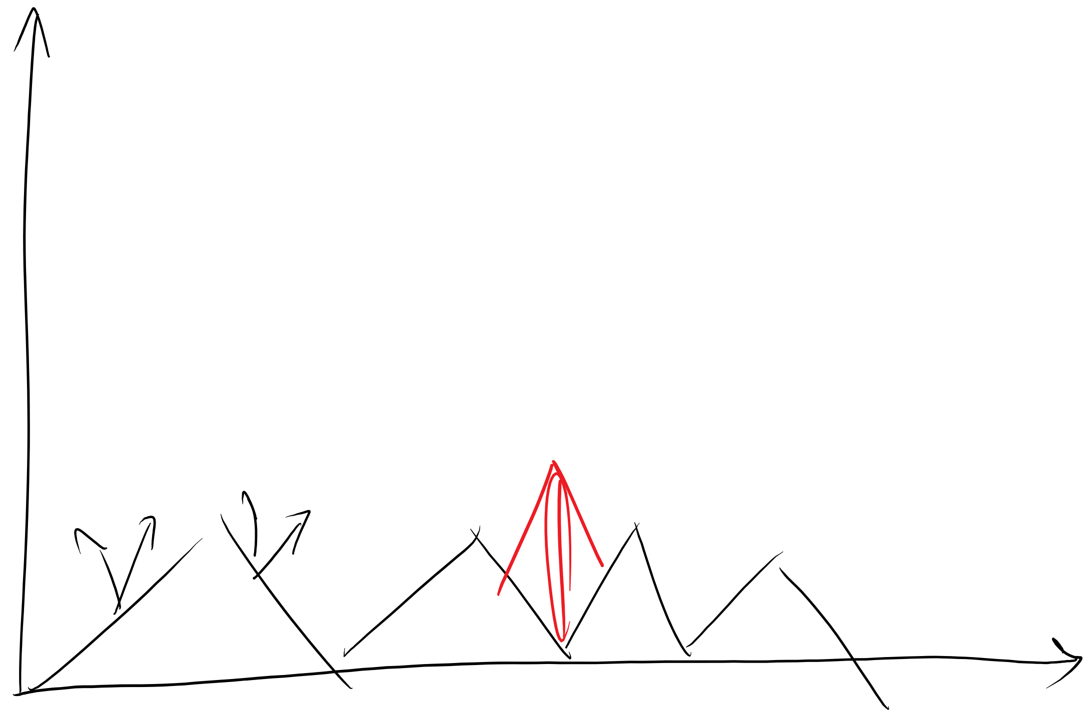
h-Adaptivity in space



h-Adaptivity in space time

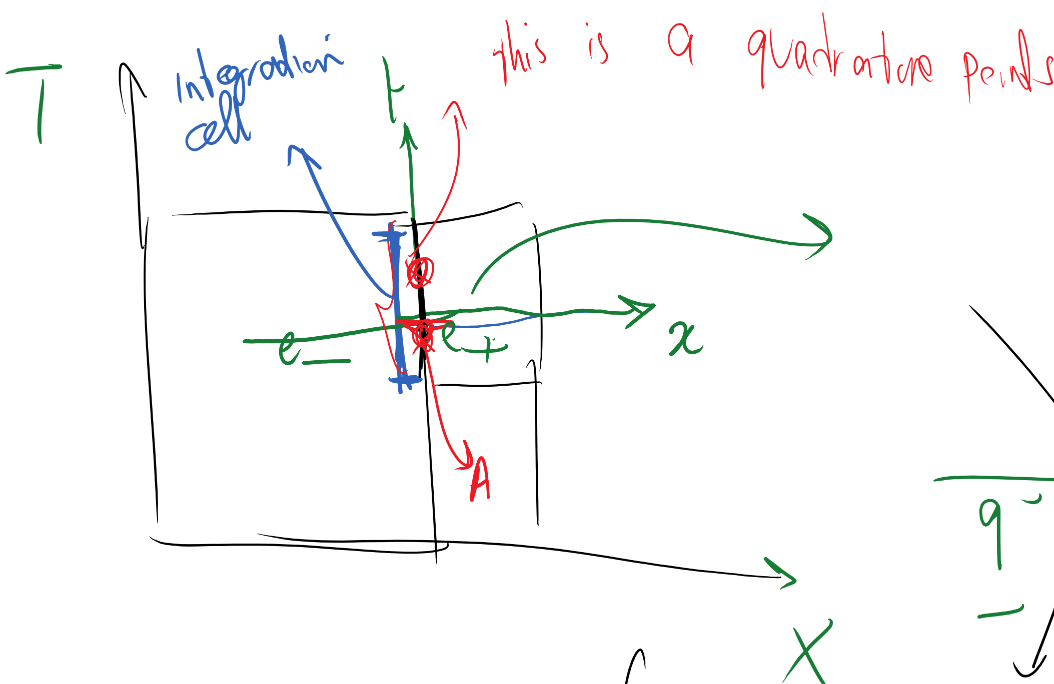


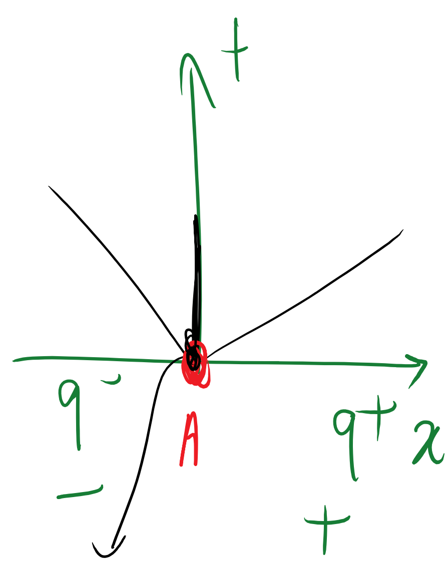
④ a SDG method Richter 1990



Discussion on Riemann solutions for (3) and (4)

1. For both having source term does not complicate things!





we get the Rieman solution on the SAME point

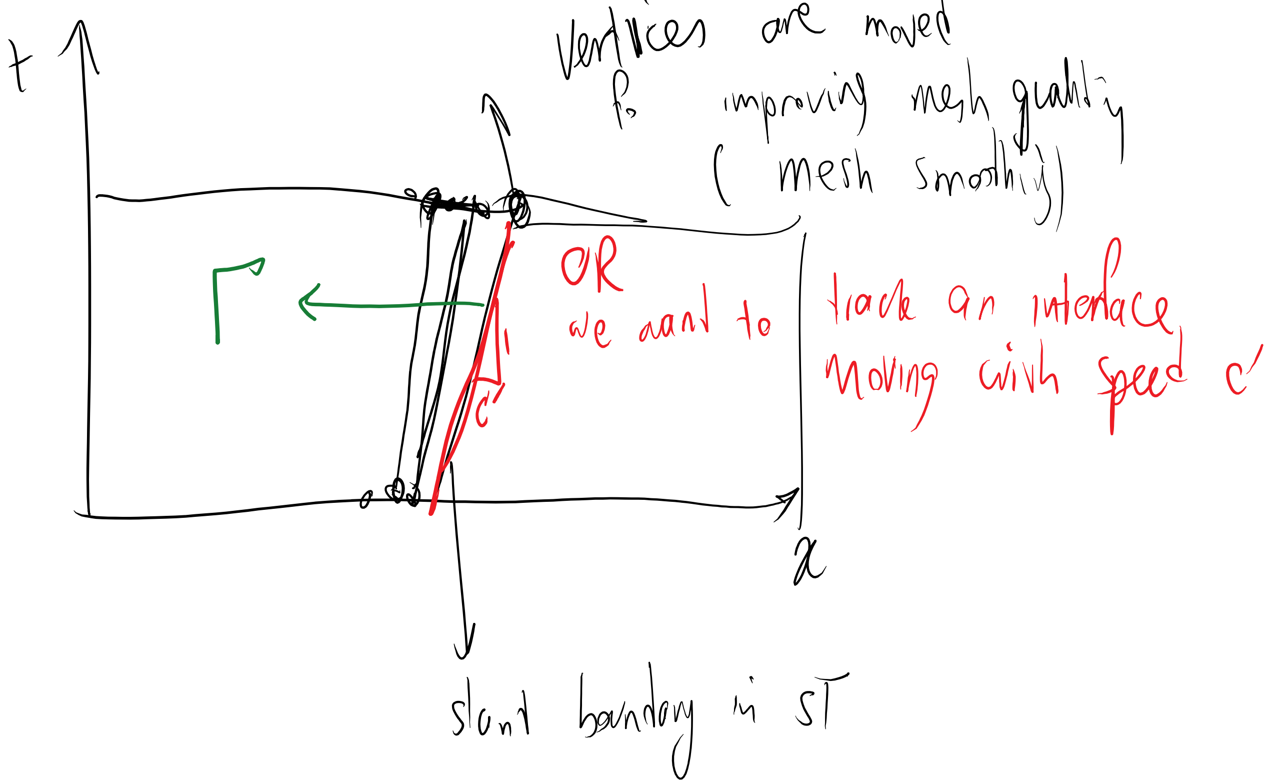
For any spacetime DG methods source terms do not influence the Riemann Solution!

terms do not influence the Riemann Solution!

solution on the SAME point

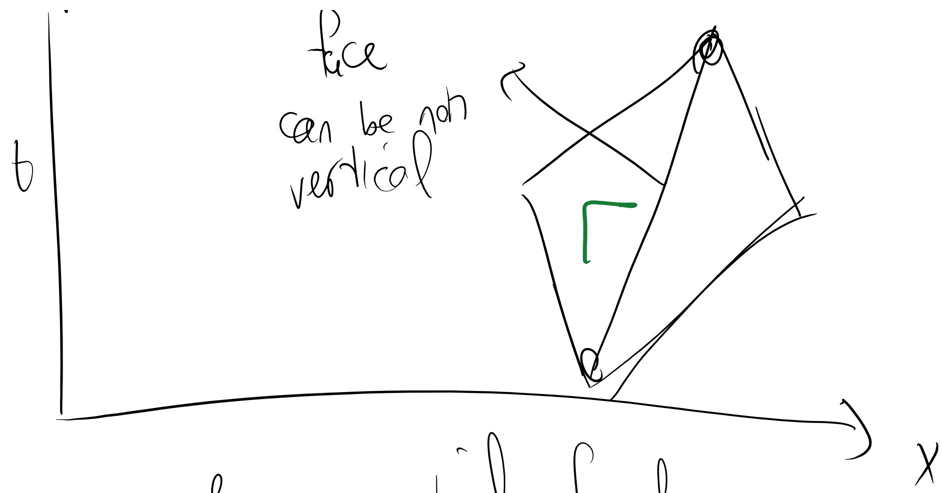
(because unlike time-marching DG methods we have the solution between time steps)

2,3. Related to fully unstructured meshes (aSDG method) OR SDG method with front tracking.



aSDG + face

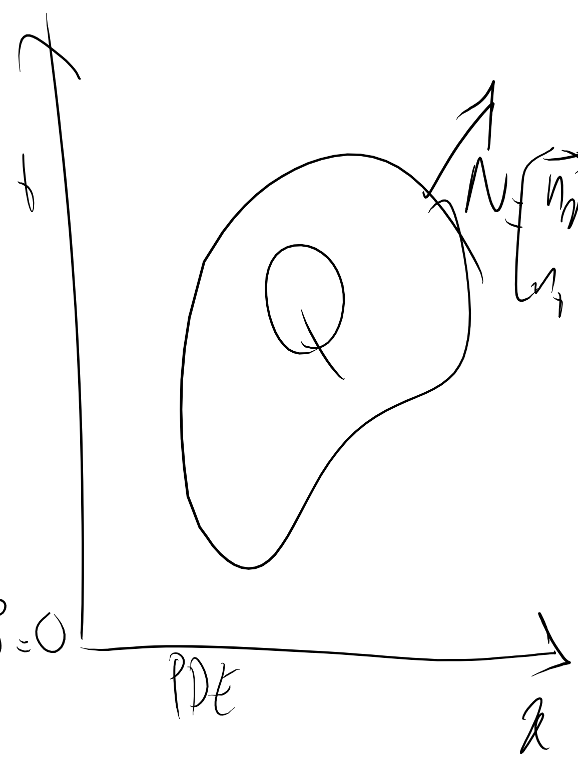
aSDG +



2.3 → Challenges of non_vertical facets

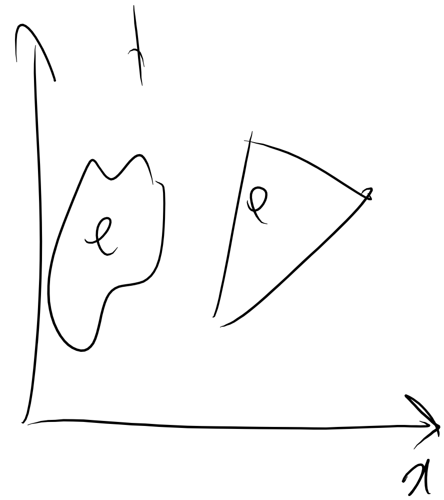
Let's revisit weighted residual statement for spacetime methods.

$$F = \begin{bmatrix} f_x \\ \hline f_t \end{bmatrix} \begin{matrix} \rightarrow \text{spatial flux} \\ \rightarrow \text{temporal flux} \end{matrix}$$



$$\int_{\partial Q} F \cdot N \, ds - \int_{Q} r \, ds = 0 \equiv \begin{cases} \nabla_{st} \cdot F - S = 0 & \text{PDE} \\ [F] \cdot N = 0 & \text{Jump part} \end{cases}$$

WRS of this

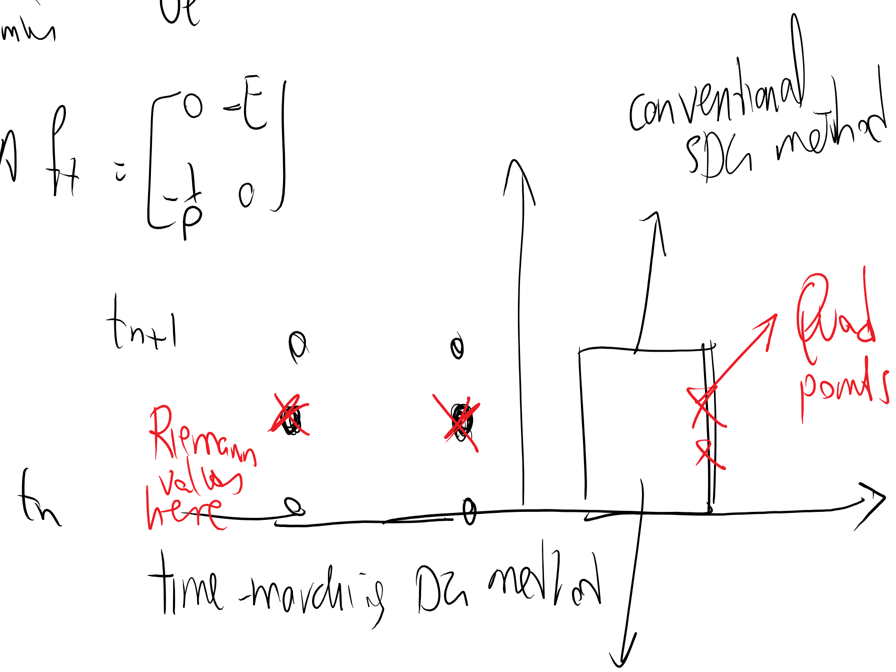


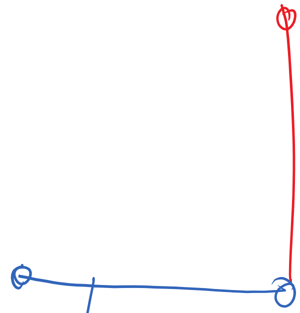
$$\int_e w\left(\nabla_{st}\cdot F - r\right) de \;+\; \int_{\partial e} w\,[F]\cdot N \,ds = 0$$

$$\int w\left(\nabla_{st}\cdot F - r\right) de + \int_{\partial e} \Big[w\left(f_x^* - f_x\right)\cdot \vec{n}_x + w\left(f_t^* - f_t\right) n_t \Big] ds = 0$$

Example: 1D Elastodynamics

$$f_t = \underbrace{\begin{bmatrix} p \\ \varepsilon \end{bmatrix}}_{q} \qquad f_x = \begin{bmatrix} -\sigma \\ -v \end{bmatrix} = A\, f_t = \begin{bmatrix} 0 & -E \\ -\frac{1}{\rho} & 0 \end{bmatrix}$$





$\int w (f_x^* - f_x) \vec{n}_x \, ds$

e.g. solid mechanics $n_t = 0$

ϵ | σ

v | p

$f_x = \begin{bmatrix} -\sigma \\ -v \end{bmatrix}$

σ^*, v^*

$\int w (f_t^* - f_t) n_t \, ds$

$n_t \to \pm 1$

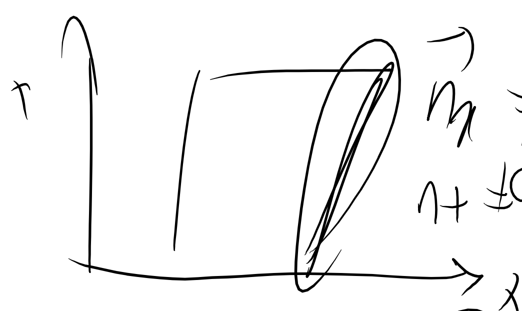
$\vec{n}_x = 0$

$f_t = \begin{bmatrix} p \\ \epsilon \end{bmatrix}$ on horizontal faces

p^*, ϵ^*

2. For non-vertical faces -> we need to compute ALL star values.

Typically Riemann solutions are only provided for fx values because faces are often vertical.



$\vec{n}_x \neq 0$

$n_t \neq 0$

f_t^* needed

f_x^*

$w((f_t^* - f_t) n_t + (f_x^* - f_x) n_x)$

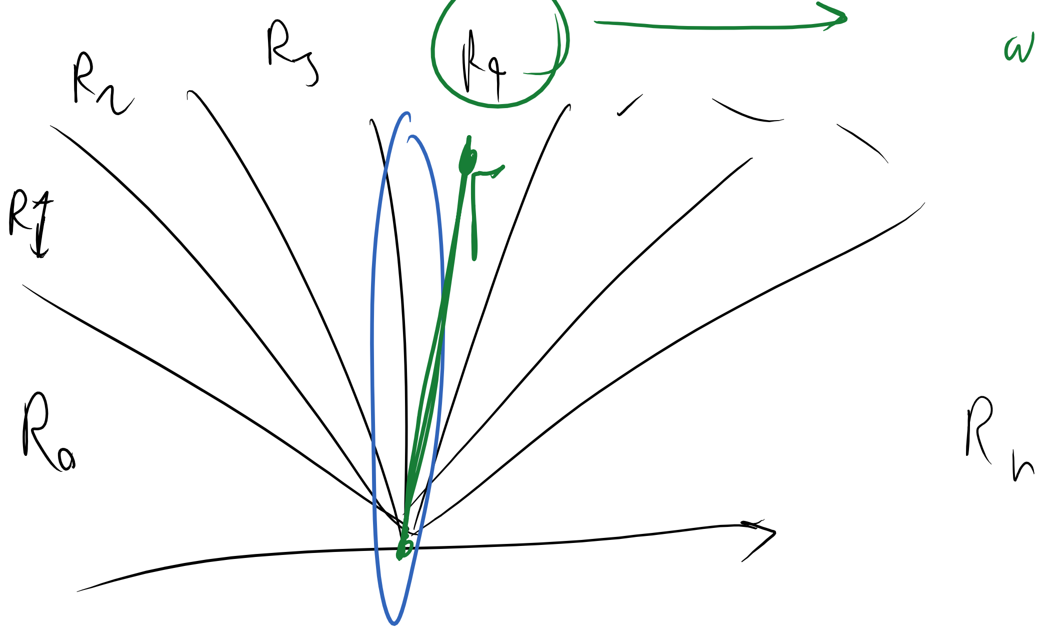
$$\omega\left(|f_t^{*} - f_t| n_t + |f_x^{*} - f_x| n_x \right)$$

Solid Mechanics

$f_x^{\theta} \longrightarrow$ add f_t^{θ}

$\sigma^{\theta}, v^{\theta} \longrightarrow$ Add $\varepsilon^{\theta}, p^{\theta}$

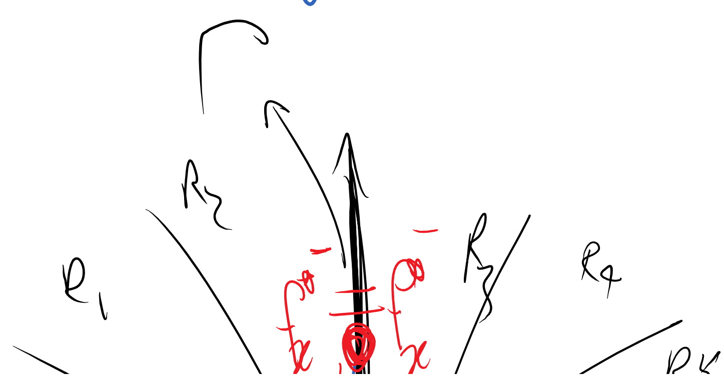
3. Values are needed on NON-Vertical faces (if the face is non-vertical)

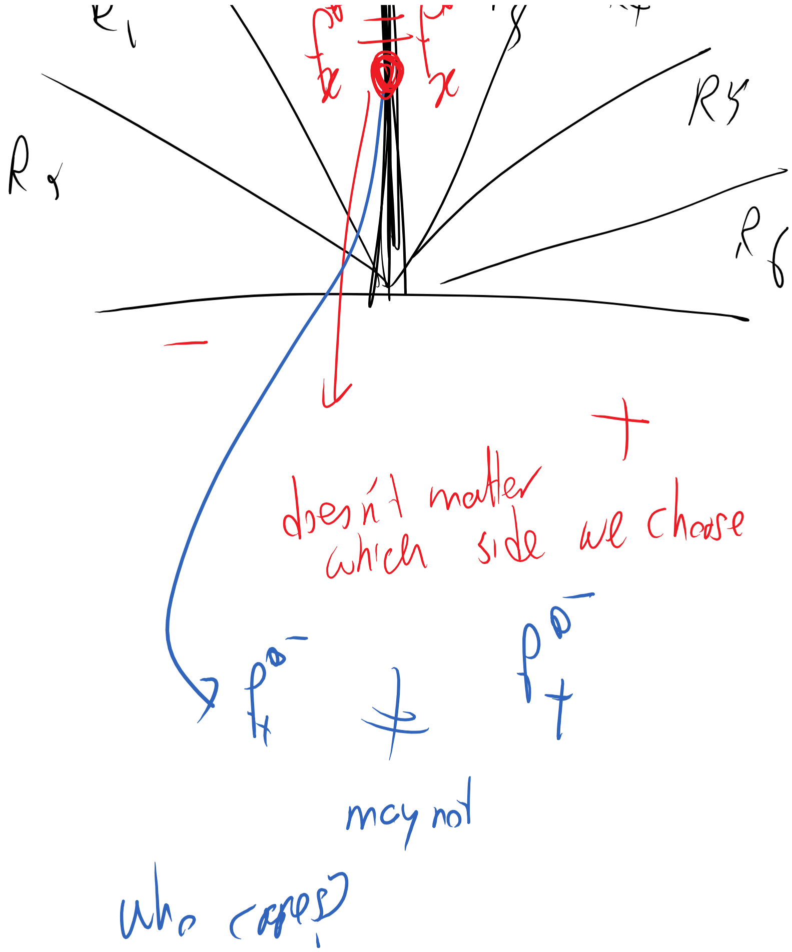


we need to choose the solution for this region

Conventional DG only needs the solution here

Trick question



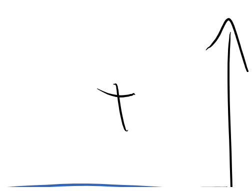


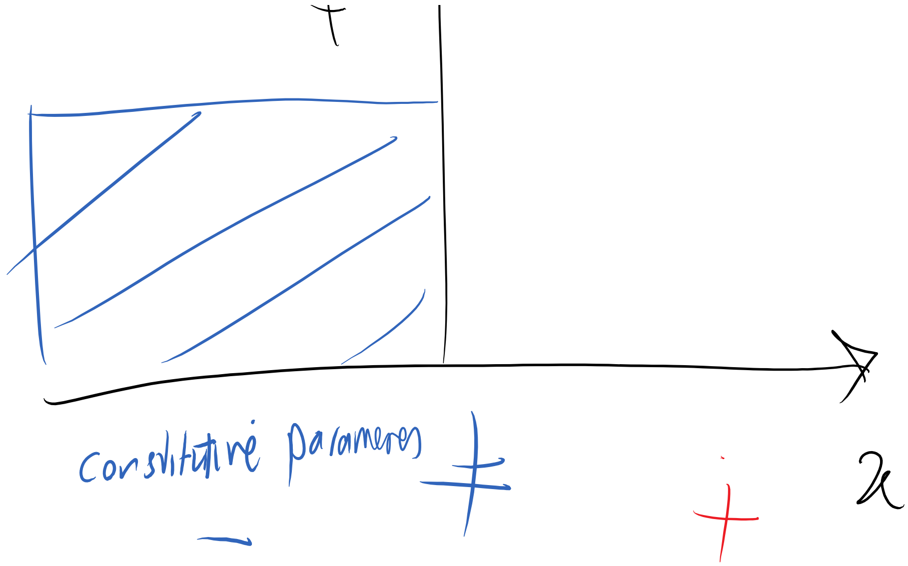
Summary:

Spacetime DG methods:

1. No need to worry about source terms when computing target (Riemann fluxes) [there are also some advantages in the stability of these methods when solving problems with high source terms].
2. Both fx* and ft* are needed in general.
3. Solutions are needed for all regions of Riemann solution.

Material Interfaces and Riemann solutions:





OR even different physics

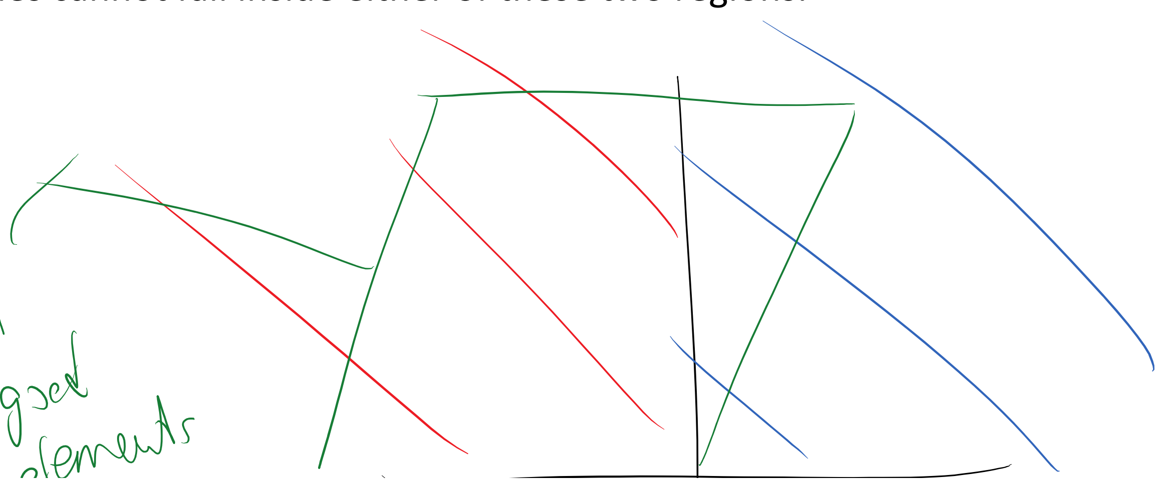
solid | fluid

Elastodynamics

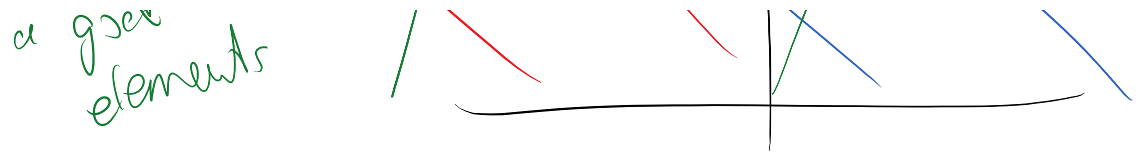
e.g. Acoustic

or more advanced

The good news is that we only need to solve the star values on the vertical direction because element boundaries cannot fall inside either of these two regions.



Not a good elements



In general the element boundary must track discontinuity (when can be nonvertical).

Example:
Solution mechanics, two different materials

Approach 1: characteristic values

$$\begin{cases} \rho \dot{p} - \sigma_{,x} = \rho b \\ \dot{\sigma} - \frac{E}{\rho} p_{,x} = 0 \end{cases}$$

$$\dot{\varepsilon} - \frac{1}{\rho} p_{,x} = 0$$

$$\dot{\varepsilon} - v_{,x} = 0$$

$$q = \begin{bmatrix} p \\ \sigma \end{bmatrix} \qquad q_{,t} + A q_{,x} = 0 \qquad A = \begin{bmatrix} 0 & -1 \\ -\frac{E}{\rho} & 0 \end{bmatrix}$$

In general we must be careful about interface matching conditions no matter how we express the system of conservation laws.

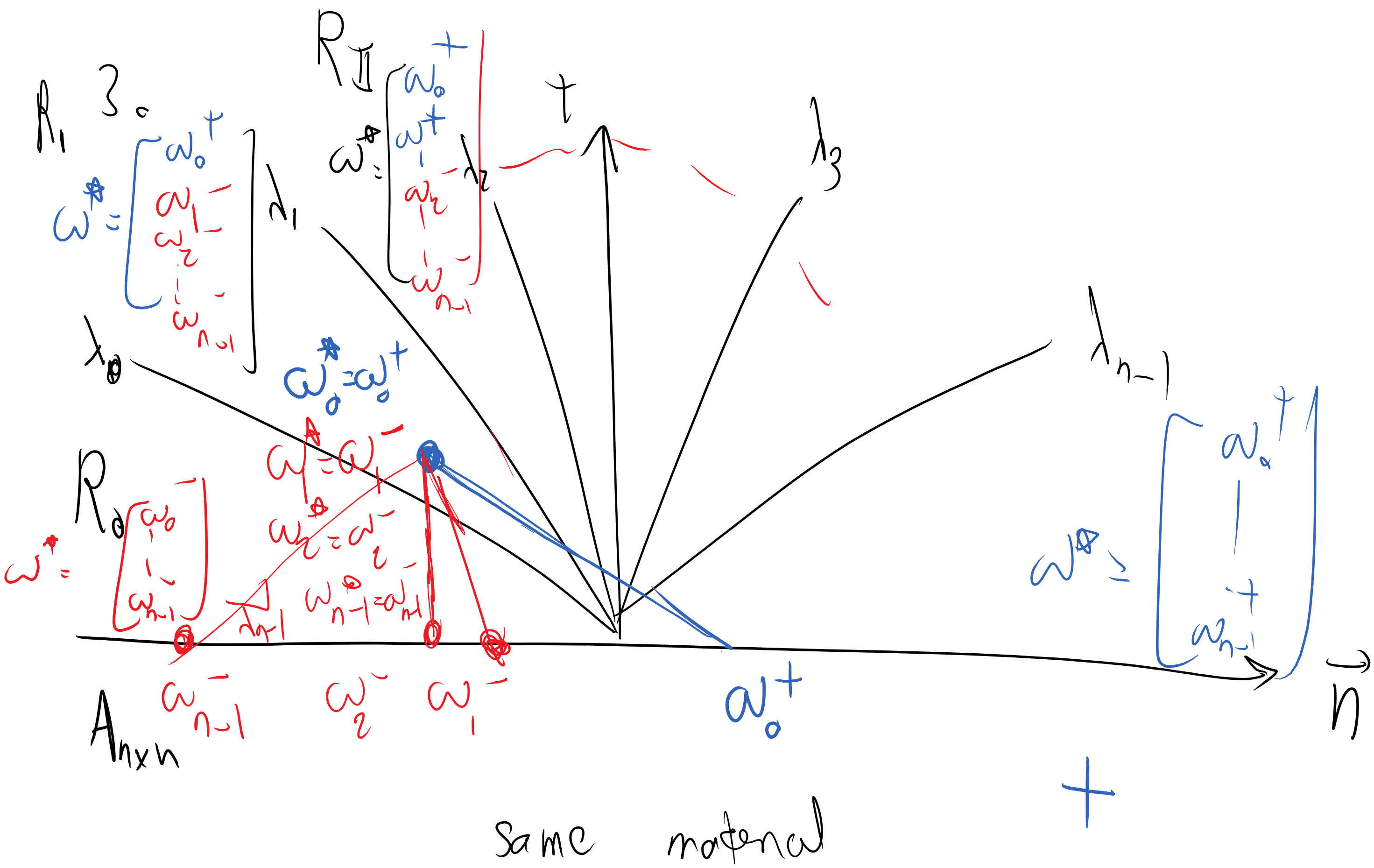
The solution scheme from the last times:

1. $W = U q$ — matrix: left eigenvectors $\begin{bmatrix} \frac{c^{(1)}}{\quad} \\ \frac{\quad}{c^{(2)}} \\ c^{(n)} \end{bmatrix}$

2. $\dot{W}_n - \lambda_n W_{n,x} = 0$

2. $\dot{w}_n - \lambda_n w_{n,x} = 0$ → eigenvalue $\forall n$

3.



3. For all regions we have ω^*

we want $f_L^* = q^* = U^{-1}\omega^*$ $\quad \omega = Uq$

& $f_x^* = Aq^* = AU^{-1}\omega^* \; UDU^{-1}\omega^*$

$= U^{-1} D \omega^{3}$