2018/04/11

Wednesday, April 11, 2018 11:40 AM

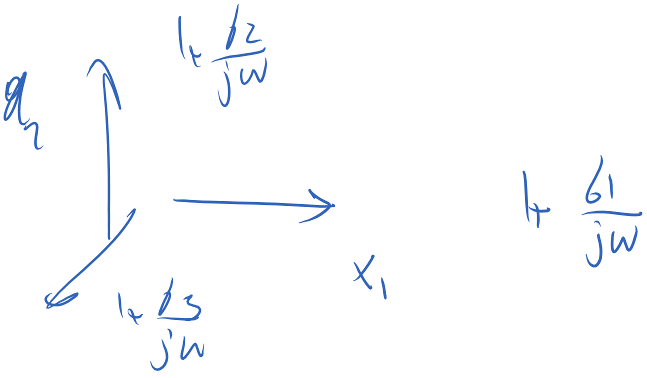
Some points on PML from last time:

direction

stretching looks like $S_i(\omega) = 1 + \frac{\sigma_i}{j\omega}$ → frequency

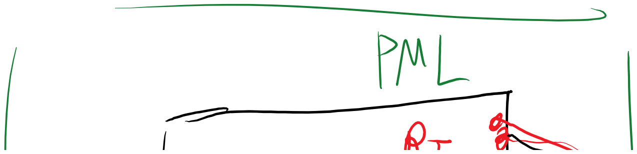
imaginary number $j^2 = 1$

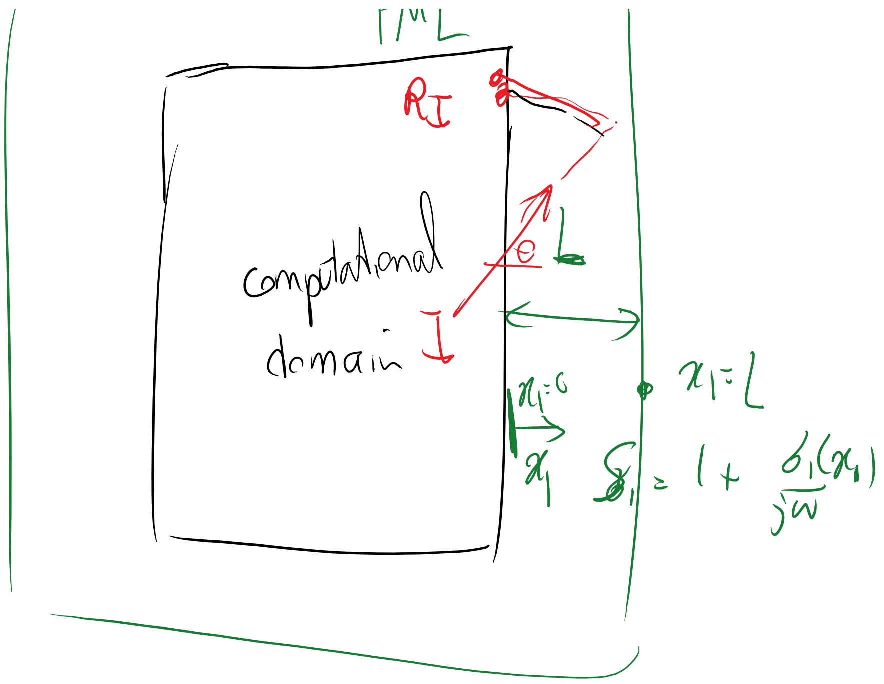
σ_i's have units of frequency



there are better stretching functions that work well for evanescent waves and for $\omega \to 0, \infty$.

- these materials are dispersive (properties depend on ω)
  - very difficult to model in time domain (TD).





$$R = \text{reflection coefficient} = \exp\left(-\frac{2\cos\theta}{\underset{\text{wave speed}}{c}}\int_0^L \sigma(x_1)\,dx_1\right)$$

$\sigma(x_1)$ constant

$$S_1 = 1 + \frac{\overset{\text{constant}}{\sigma_1}}{j\omega}$$

$$\boxed{R = \exp\left(-2\cos\theta\,\frac{L\sigma_1}{c}\right)}$$

$\theta \to \frac{\pi}{2}$ worst

$\theta \to 0$ the best

$L \to \infty$ $R \to 0$ ✓

$\sigma_1 \to \infty$ $R \to 0$

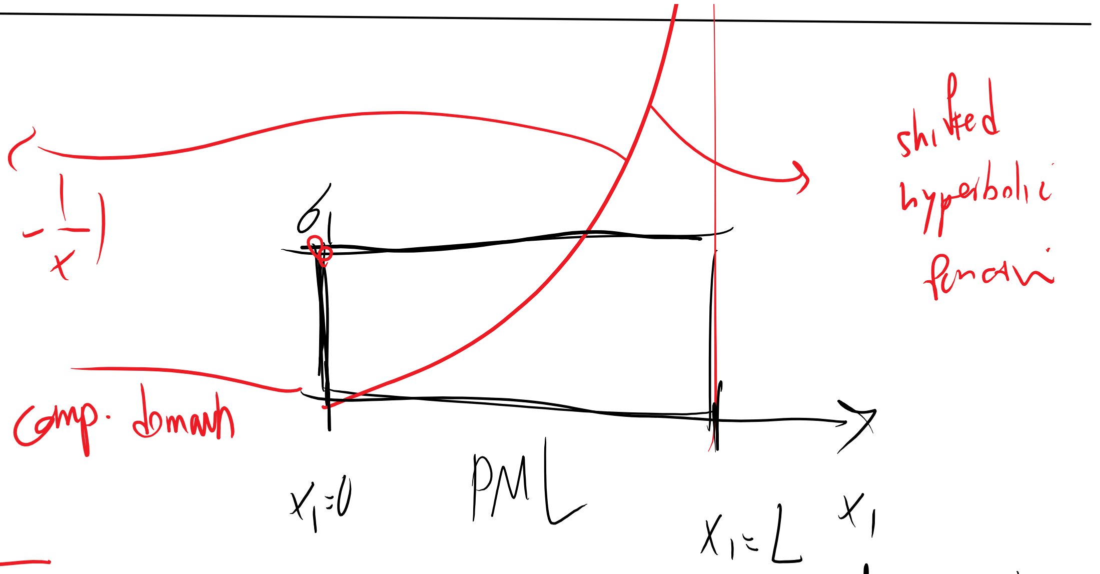
$\sigma_1(x)$

$S_1(\omega) = 1 + \frac{\sigma_1(x)}{j\omega}$

$\sigma_1(x) = \sigma^0\left(\frac{1}{L-x} - \frac{1}{x}\right)$

$\sigma_1(0) = 0$

$\sigma_1(L) \to \infty$



$$\boxed{R \to 0}$$

benefits
1: No sudden material property change
2. $\sigma(x \to L) \to \infty \Rightarrow R \to 0$

Continuum aspects

Even if two materials have matching impedances →
numerically we get some reflection

$Z^- = Z^+$

but have some numerical reflection! Guddati!

that's where property ① becomes useful.

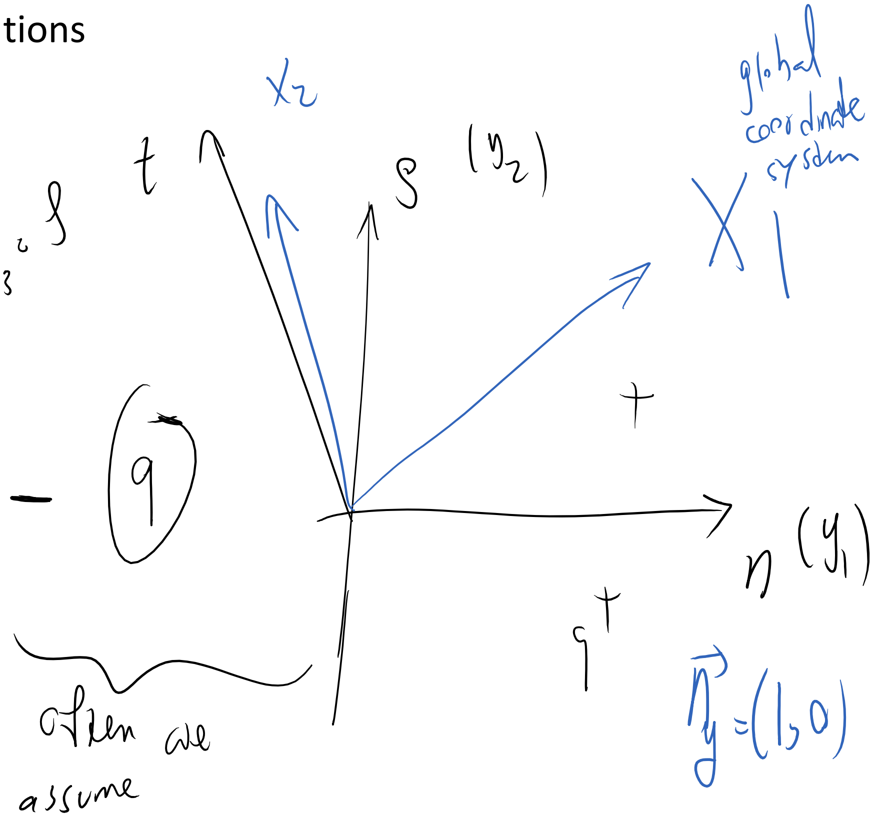
In DG methods the continuity of material property is not as important and in fact some studies show that unlike CFEM, FD, a sudden jump in sigma_1 (constant value) may still result in better numerical results.

Going back to 2D, 3D, Riemann solutions

2D x time

$$\dot{f}_t + f_{x_1,x_1} + f_{x_2,x_2} + f_{x_3,x_3} = S$$

$$\dot{f}_t \downarrow q$$



The best way is to solve this in the local coordinate system

often we assume that on each side we have constant value

$$\dot{f}_t + f_{y_1,y_1} + \boxed{f_{y_2,y_2} + f_{y_3,y_3}} = S$$

these quantities are zero because of IC set up

$n_y = (1, 0, 0)$

↳ spatial normal

$f^* \cdot n =$

$f^*_{y_1} n_{y_1} + f^*_{y_2} \underbrace{\cancel{n_{y_2}}}_{0} + f^*_{y_3} \underbrace{\cancel{n_{y_3}}}_{0}$

$= \boxed{f^*_{y_1}} n_{y_1}$

that's all we need

Benefits of sln in local coordinate:

① Right away get rid of f_{y_2}, f_{y_3}

② we only get what's needed $f^*_{y_1}$

To demonstrate this better, let's consider a <u>linear</u> problem

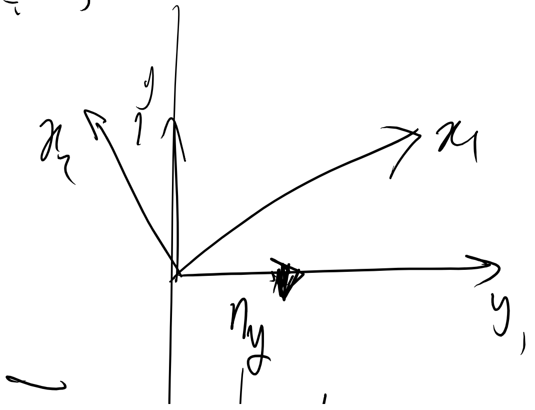
f_t

$$\dot{q} + A_1^x q_{,x_1} + A_2^x q_{,x_2} + A_3^x q_{,x_3} = f$$

from top

$$q_{,y_1} = A_1^{x} \overset{\uparrow \text{x system}}{(n_y^x)_1} + A_2^x (n_y^x)_2 + A_3^x (n_y^x)_3$$

$$= A_n$$



$= A_n$

$\sim \Big| \; n_y \; + \; y_1$

$\mathring{q} + A_n \, q_{,y_1} = S$

expressed in $\underline{x}$ system

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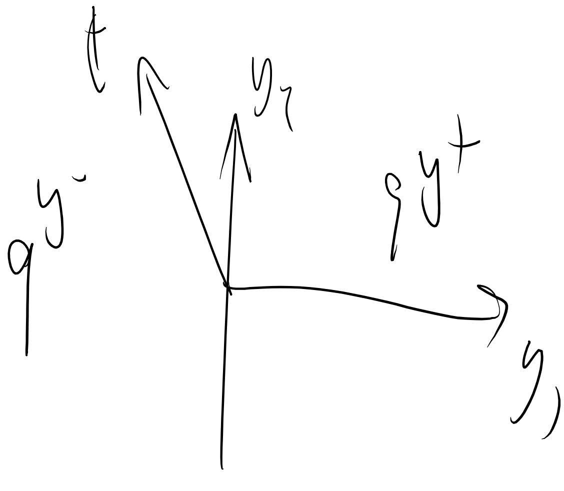
typical routine for star value calculation

① q^{x^-}, q^{x^+} values are computed in global x coordinate system

$\downarrow$

② use n_y^x (normal vector n_y expressed in x system) to rotate components of q^{x^-}, q^{x^+} to y coordinate system

q^{y^-}, q^{y^+}, Compute A_1 matrix as well



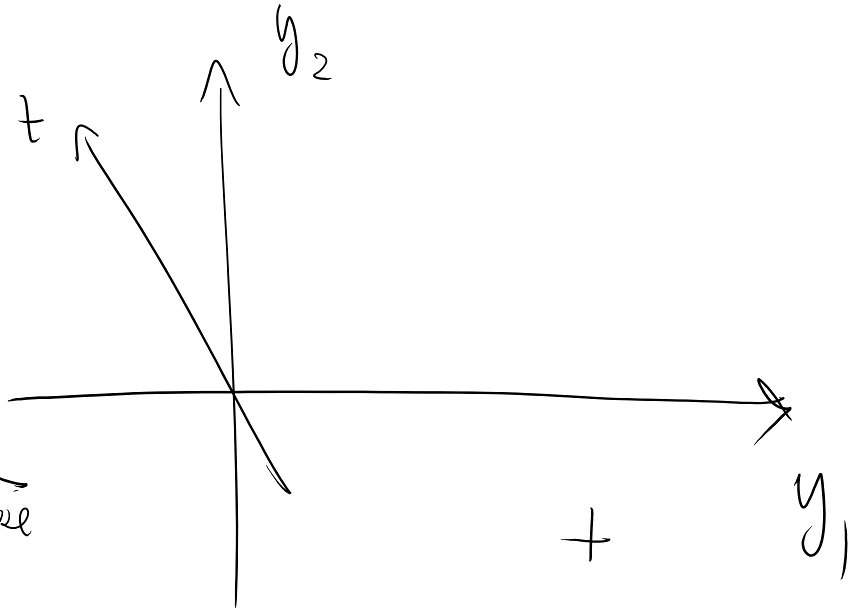
③ Solve the Riemann problem

temporal flux $\quad (q^*_y)^- \qquad (q^*_y)^+$

spatial flux $\quad (A_1 q^*_y)^- \qquad (A_1 q^*_y)^+$

(3) Most often the WRS is objective & can be expressed in any coordinate system including y, so we directly compute WRS (WK) in this system

Example: 3D elastodynamics:



kinematic $\qquad$ force-like

$V_{3\times1} \longleftrightarrow P_{3\times1} = \rho V$

$\epsilon_{3\times3}$
symmetric
tensor

$$\epsilon_{3\times3} \longleftrightarrow \sigma_{3\times3} = C_{3\times3\times3\times3}\, \epsilon_{3\times3}$$

$\downarrow$

Using Voight notation

$$\gamma = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \end{bmatrix} \qquad S = \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{31} \end{bmatrix}$$

$$S_{6\times1} = \bar{C}_{6\times6}\ \gamma_{6\times1}$$

$\downarrow$ Voight stiffness

$$\gamma_{6\times1} = D_{6\times6}\ S_{6\times1}$$

$\downarrow$ compliance

$$f_n = \begin{bmatrix} \epsilon S \\ \tilde{\nabla} V \end{bmatrix}$$

$$V_{3\times1} \longleftrightarrow P_{3\times1} = \rho V$$

$$\gamma_{6\times1} \longleftrightarrow S_{6\times1} = \bar{C}_{6\times6}\ \gamma$$

$$\begin{pmatrix} \dot{P} \\ \dot{\epsilon} \end{pmatrix} - \begin{pmatrix} \nabla \cdot \sigma = f \\ \nabla V = 0 \end{pmatrix}$$

$_{6\times1} \longleftrightarrow s_{6\times1} = C_{6\times6}\,\gamma$

$\underbrace{\quad} - \underbrace{\nabla V} = 0$

ρ

$\gamma \rightarrow$ should go to q vector

what if we want to make our lives easier in terms of interface matching conditions

I prefer working with spatial flux quantities put in q as 1) in case of material property jumps the solution on vertical interface does not jump [fy] = 0), 2) we need spatial fluxes on vertical faces anyway (for most DGs).

$$q = \begin{array}{r} \\ v \longrightarrow \\ \\ \\ \\ \\ s_{6\times1} \longrightarrow \\ \\ \\ \end{array}\left[\begin{array}{c} v_1\\ v_2\\ v_3\\ \hline s_{11}\\ s_{22}\\ s_{33}\\ s_{12}\\ s_{23}\\ s_{31}\end{array}\right]$$

6 eqns $\quad \dot{p} - \nabla\cdot\sigma = \rho b \quad$ 3 eqns

$\frac{1}{\rho} \quad v \quad \longleftarrow$

$\dot{\varepsilon}_{3\times3} - \nabla V = 0 \quad$ 6 eqns

$C \quad \times \quad \gamma \longleftarrow$

$$\begin{cases} \mathring{v} - \frac{1}{\rho} \nabla \cdot \sigma = b \\ \mathring{\sigma} - \underset{3\times3\times3\times3}{C}(\nabla v) = 0 \end{cases}$$

2a

3 eqns

$$\boxed{\begin{aligned} \mathring{v}_1 &- \frac{1}{\rho}\left(\overset{s_1}{\uparrow}\!\sigma_{11,1} + \overset{s_4}{\uparrow}\!\sigma_{12,2} + \overset{s_6}{\uparrow}\!\sigma_{13,3} \right) = b_1 \\ \mathring{v}_2 &- \frac{1}{\rho}\left(\overset{s_4}{\uparrow}\!\sigma_{21,1} + \overset{s_2}{\uparrow}\!\sigma_{22,2} + \overset{s_5}{\uparrow}\!\sigma_{23,3} \right) = b_2 \\ \mathring{v}_3 &- \frac{1}{\rho}\left(\overset{s_6}{\uparrow}\!\sigma_{31,1} + \overset{s_5}{\uparrow}\!\sigma_{32,2} + \overset{s_3}{\uparrow}\!\sigma_{33,3} \right) = b_3 \end{aligned}}$$

$$S = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}$$

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = D_{6\times6} \begin{bmatrix} \overset{\gamma}{\uparrow}\epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \end{bmatrix} \begin{array}{l} \to u_{1,1} \\ \to u_{2,2} \\ \to u_{3,3} \\ \to (u_{1,2} + u_{2,1}) \\ \to (u_{2,3} + u_{3,2}) \\ \to (u_{3,1} + u_{1,3}) \end{array}$$

take the time derivative

$$\mathring{S} - D \begin{bmatrix} v_{1,1} \\ v_{2,2} \\ v_{3,3} \\ v_{1,2} + v_{2,1} \\ v_{2,3} + v_{3,2} \end{bmatrix} = 0$$

2b

6 eqns

$$\boxed{\begin{pmatrix} \vdots \\ v_{2,3}+v_{3,2} \\ v_{3,1}+v_{1,3} \end{pmatrix}}$$

9 eqns

for example 1st eqn for 2D

$$\dot{S}_1 - C_{11} v_{1,1} - D_{12} v_{2,2} - D_{13} v_{3,3} - D_{14}(v_{1,2}+v_{2,1}) - D_{15}(v_{2,3}+v_{3,2}) - D_{16}(v_{3,1}+v_{1,3}) = 0$$

,2 derivatives ,1 derivatives ,3 derivatives

& 4th one

$$\dot{S}_4 - S_{41} v_{1,1} - D_{42} v_{2,2} - D_{43} v_{3,3} - S_{44}(v_{1,2}+v_{2,1}) - D_{45}(v_{2,3}+v_{3,2}) - S_{46}(v_{3,1}+v_{1,3}) = 0$$

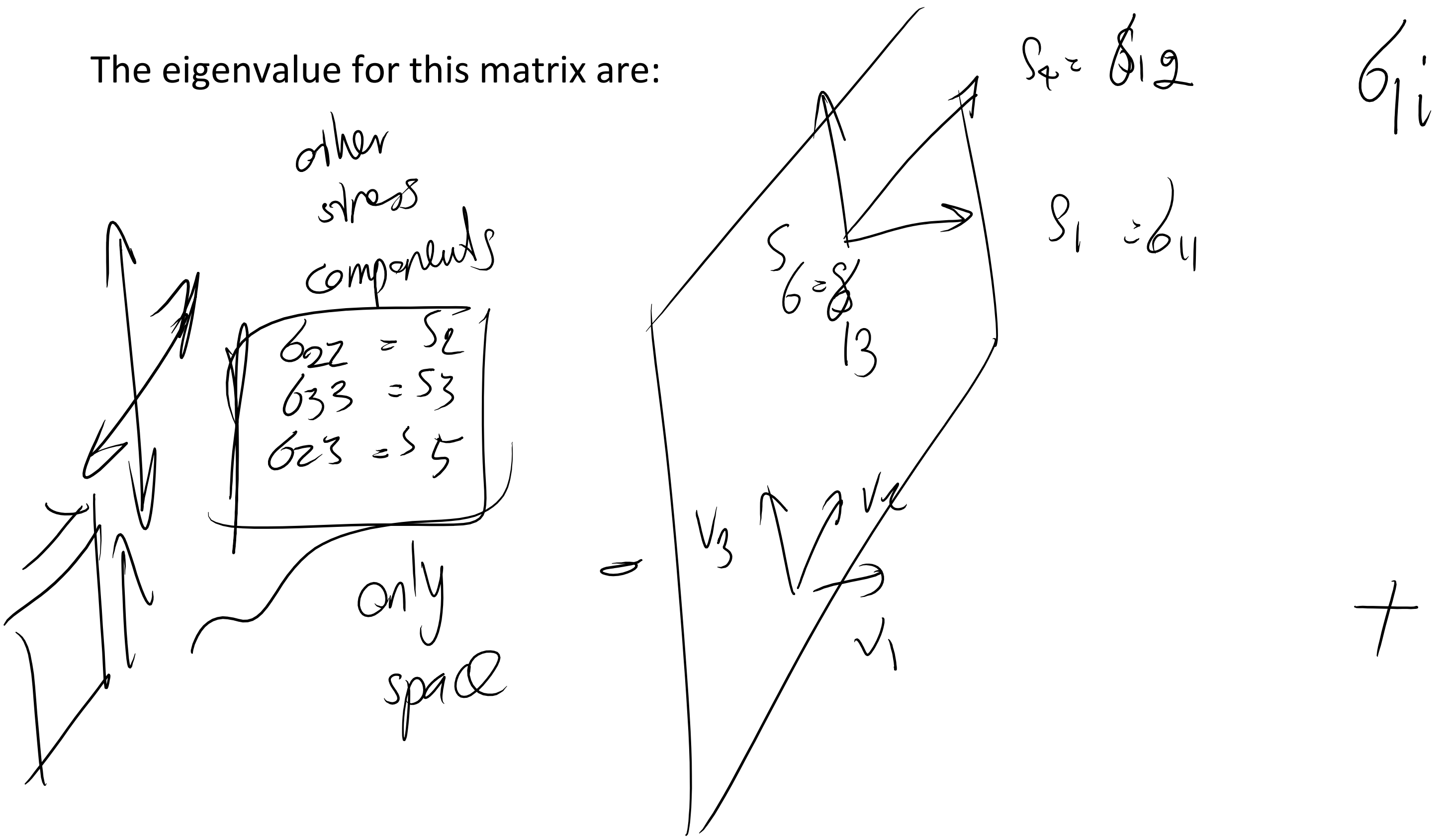
$$\dot{q} + \cdots = 0 \qquad q = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ s_6 \end{pmatrix}$$

$$\dot{q} + A_1 q_{,1} + A_{2,2} + A_3 q_{,3} = 0$$

$$A = \begin{array}{c|ccccccccc} & V_{1,1} & V_{2,1} & V_{3,1} & S_{1,1} & S_{2,1} & S_{3,1} & S_{4,1} & S_{5,1} & S_{6,1} \\ \hline \dot{V}_1 & & & & -\beta & & & & & \\ \dot{V}_2 & & & & & & & -\beta & & \\ \dot{V}_3 & & & & & & & & & -\beta \\ \dot{S}_1 & C_{11} & C_{14} & C_{16} & & & & & & \\ \dot{S}_2 & C_{21} & C_{24} & C_{26} & & & & & & \\ \dot{S}_3 & C_{31} & C_{34} & C_{36} & & & & & & \\ \dot{S}_4 & C_{41} & C_{44} & C_{46} & & & & & & \\ \dot{S}_5 & C_{51} & C_{54} & C_{56} & & & & & & \\ \dot{S}_6 & C_{61} & C_{64} & C_{66} & & & & & & \end{array}$$

A_1

The eigenvalue for this matrix are:



V_1, V_2, V_3 are continuous

$\sigma_{11}, \sigma_{12}, \sigma_{13}$

S_1, S_4, S_6 are continuous

$\sigma_{22}, \sigma_{33}, \sigma_{23}$ can suffer jump