2022/09/20

Tuesday, September 20, 2022 9:39 AM

There are many options for the star values.
For now, we are only going to use "Average" flux option.

Star values for T:

e^- T_{e^-} T_{e^+} e^+

average flux $T^* = \dfrac{T_{e^-} + T_{e^+}}{2}$

$T^{e_1} = a_1 + a_2 x^{(1)}$ $\quad T^{e_2} = a_3 + a_4 x^{(2)}$ $\quad T^{e_3} = a_5 + a_6 x^{(3)}$

$T^{e_1} = a_1$ $\quad T^{e_1} = a_1 + a_2$ $\quad T^{e_2} = a_3$ $\quad T^{e_2} = a_3 + a_4$ $\quad T^{e_3} = a_5$ $\quad T^{e_3} = a_5 + a_6$

∂D_u $\quad T = \bar{T}$ $\quad\quad \partial D_f$ $\quad \bar{q} = 1$

$T^* = \bar{T}$ ∂D_u $\quad T^{*R}_{e_1} = T^{*L}_{e_2} = \left(\dfrac{a_1 + a_2 + a_3}{2}\right)$ $\quad T^{*R}_{e_2} = T^{*L}_{e_3} = \left(\dfrac{a_3 + a_4 + a_5}{2}\right)$ $\quad T^{*R}_{e_3} = a_5 + a_6$

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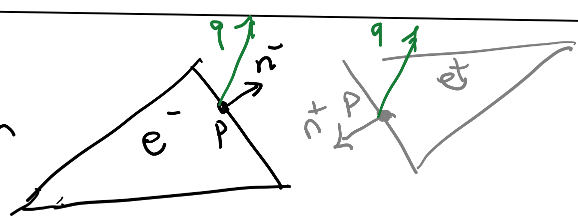
q^*_n values



Balance law we have the term

$\int_{\partial\Omega} q \cdot n \, ds \longrightarrow$ Jump condition $\quad [q] \cdot n = 0 \rightarrow (q^+ - q^-) \cdot n^- = 0$

subdomain

$q^+ n^- = q^- n^- \rightarrow -\underbrace{(q^+ n^+)}_{\text{net flux going out @ P in element } e^+} = \underbrace{q^- n^-}_{\text{" " @ P in element } e^-}$

$\boxed{q_{n^+} = -q_{n^-}}$

similar to action-reaction with traction
$t^+ = -t^-$
$\sigma n^+ = -\sigma n^-$

$$\boxed{q^{*+}_n = -q^{*-}_n}$$

So, the approach to compute the star value for spatial fluxes is:

1) $q^* = f(q^{\mp}, T^{\mp})$ vector

2) $q^{*+}_n = q^* \cdot n^+, \quad q^{*-}_n = q^* \cdot n^-$ scalars

Again, one choice of q^* is $\quad \boxed{q^* = \left(\dfrac{q^- + q^+}{2}\right)}$ average flux

$T = a_1 + a_2 x^{(1)}$ $\quad T^{e_2} = a_3 + a_4 x^{(2)}$ $\quad q^{e_3} = -a_6$

$T = a_1 + a_2 x^{(1)}$

$q = -k a_2 = -a_2$

$n = -1$ $\qquad$ $n = 1$

$\partial\Omega_a (T = \bar{T})$ $\qquad$ $(q^{e_1})^R = -a_2$

$q^* = q = -a_2$

$q^*_n = (-a_2)(-1) = a_2$

$T^{e_2} = a_3 + a_4 x^{(2)}$

$q^{e_2} = -a_4$

-1 $\qquad$ $+1$

$(q^{e_2})^L = -a_4$ $\qquad$ $(q^{e_2})^R = -a_4$

$$q^* = \frac{(q^{e_1})^R + (q^{e_2})^L}{2} = -\left(\frac{a_2 + a_4}{2}\right)$$

$$(q^*_n)^{e_1}_R = q^*(1) = -\left(\frac{a_2 + a_4}{2}\right)$$

$$q^{*e_2}_{n_L} = q^*(-1) = \frac{a_2 + a_4}{2}$$

$$q^{*e_2}_{n_R} = -\left(\frac{a_4 + a_6}{2}\right)$$

$q^{e_3} = -a_6$

e_3

-1 $\qquad$ 1

$q^L_{e_3} = -a_6$ $\qquad$ $q^R_{e_3} = -a_6$

$\partial\Omega_f$

$\bar{q}$

$$q^{*e_3}_{n_R} = \bar{q} = 1$$

Plug q*n and T* into the last equation from last class to obtain:

$$M_{6\times 6}\,\dot{a} + K_{6\times 6}\,a = F$$

$$K = \begin{array}{c|c} & \begin{matrix} e_1 & & e_2 & & e_3 & \end{matrix} \\ \hline \begin{matrix} e_1 \\ \\ e_2 \\ \\ e_3 \\ \\ \end{matrix} & \begin{bmatrix} 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \end{array}, \quad F = \begin{array}{c} e_1 \\ \\ e_2 \\ \\ e_3 \\ \\ \end{array} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} \qquad \text{I(a)}$$

skin weights

The off-diagonal blocks are the coupling terms from the star values

Recall

$$m^e = \int_e N^{e^T} C_v N^e \, dv = \int_0^1 \begin{bmatrix} 1 \\ \xi \end{bmatrix} C_v \begin{bmatrix} 1 & \xi \end{bmatrix} (\underbrace{h\,d\xi}_{dx})$$

h, x_0, $\xi = \frac{x - x_0}{h}$

$$m^e = C_v h \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$ we derived this before

Use this mass matrix for DG set-up below

$h = 1$

$T^{e_1} = a_1 + a_2 x^{(1)}$ $\qquad$ $T^{e_2} = a_3 + a_4 x^{(2)}$ $\qquad$ $T^{e_3} = a_5 + a_6 x^{(3)}$

e_1 $\qquad$ e_2 $\qquad$ e_3

$\mathcal{M}^{e_1} = [1, 2]$ $\qquad$ $\mathcal{M}^{e_2} = [3, 4]$ $\qquad$ $\mathcal{M}^{e_3} = [5, 6]$

Assembling element mass matrices to the global mass (capacitance) matrix, we obtain:

$$M_{6\times6} = \begin{bmatrix} 1 & \frac{1}{2} & & & & \\ \frac{1}{2} & \frac{1}{3} & & & & \\ & & 1 & \frac{1}{2} & & \\ & & \frac{1}{3} & \frac{1}{3} & & \\ & & & & 1 & \frac{1}{2} \\ & & & & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

$$M\dot{a} + Ka = F$$

with IC $\quad a_0 = \bar{a}$ initial dofs

K & F in eq 5a

Ib

semi-discrete form for DG, 3 element problem

$x \rightarrow$ discretized

$t \rightarrow$ is not

There are many ways to discretize this in time. We'll only consider simple backward and forward Euler discretizations.

A) Backward Euler method is which is an implicit time marching scheme:

In backward Euler method, the equation is written for the current (n + 1) time step:

$$\left.\begin{aligned} &M\dot{a}_{n+1} + Ka_{n+1} = F_{n+1} \\ &\text{Backward difference for } \dot{a}_{n+1} = \frac{a_{n+1} - a_n}{\Delta t} \end{aligned}\right\} \rightarrow M(a_{n+1} - a_n) + K\Delta t\, a_{n+1} = \Delta t\, F_n$$

a_n is known, want to obtain a_{n+1}

$$\boxed{\begin{aligned} &(M + \Delta t K)\, a_{n+1} = \Delta t\, F_{n+1} + M a_n \\ &\mathcal{M}\, a_{n+1} = \mathcal{F}_{n+1} \end{aligned}}$$

IIa Implicit, using backward Euler method

B) Explicit forward method: We'll write the equation for the previous time step:

$$\left.\begin{aligned} &M\dot{a}_n + Ka_n = F_n \\ &\text{Forward E. } \dot{a}_n = \frac{a_{n+1} - a_n}{\Delta t} \end{aligned}\right\} \rightarrow M(a_{n+1} - a_n) + \Delta t\, K a_n = \Delta t\, F_n$$

a_{n+1} t_{n+1}

$\dot{a}_n = \frac{a_{n+1} - a_n}{\Delta t}$

a_n t_n

$$\boxed{\begin{aligned} &M a_{n+1} = \underbrace{\Delta t F_n + (M - \Delta t K) a_n} \\ &\mathcal{M} a_{n+1} = \mathcal{F}_{n+1} \end{aligned}}$$

IIb Explicit, using forward Euler method

Main differences:

1. Stiffness matrix contributions are on the LHS for the implicit method:
   a. As we will see this greatly complicates the structure of effective M, and makes it much worse for implicit schemes even if th e PDE is linear.
   b. If the problem was nonlinear $M\dot{a} + f^{NL}(a) = F$, in this case $K(a) = \frac{df^{NL}}{da}$ is a non -constant function. The problem remains nonlinear with implicit methods but will become linear with explicit ones.

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b. If the problem was nonlinear $M\dot{a}+f^{NL}(a)=F$, in this case $K(a)=\frac{df^{NL}}{da}$ is a non -constant function. The problem remains nonlinear with implicit methods but will become linear with explicit ones.

$$M\dot{a}+f^{NL}(a)=F$$

B.E. $M\dot{a}_{n+1}+\underline{f^{NL}(a_{n+1})}=F_{n+1}$

$$M\left(\frac{a_{n+1}-a_n}{\Delta t}\right)+\underbrace{f^{NL}(a_{n+1})}_{\text{nonlinear}}\ \ldots=F_{n+1}$$

... need to use Newton-Raphson, ... to solve this

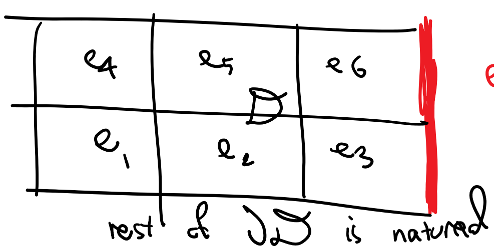
F.E $M\dot{a}_n+\underbrace{f^{NL}(a_n)}_{\text{prev. step}}=F_n$

$$M\left(\frac{a_{n+1}-a_n}{\Delta t}\right)=F_n-f^{NL}(a_n)\ \rightarrow\ Ma_{n+1}=Ma_n+\Delta t\left(F_n-f^{NL}(a_n)\right)$$

With an explicit method, we always solve a linear system.

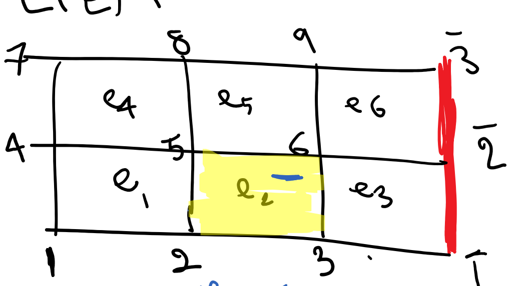
For point a, we will see that DG methods have a very "nice" system matrix $\mathcal{M}$, but for CFEM we need to use mass lumping ...

Why DG methods have an inherent advantage for **explicit** solution schemes.

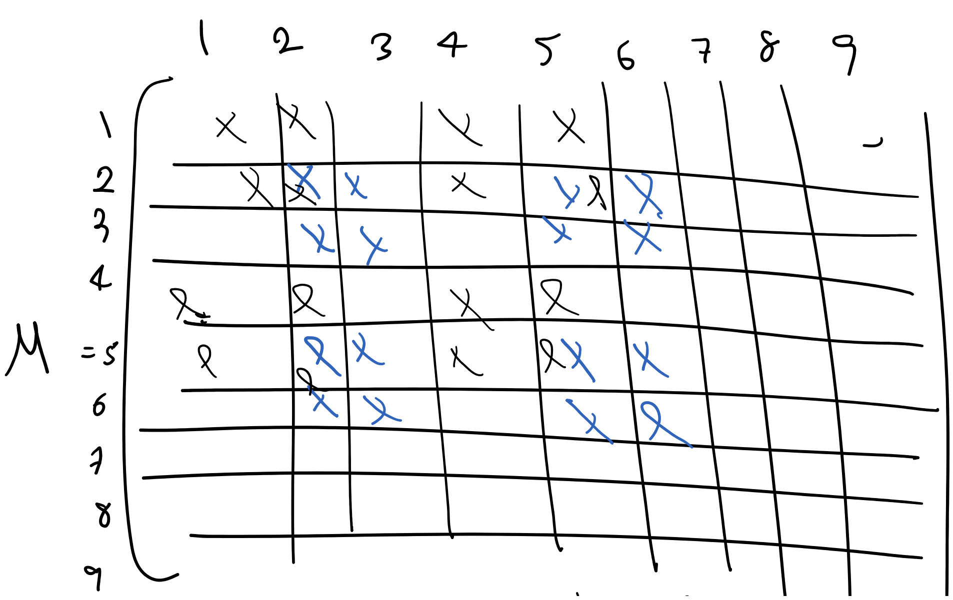


Heat conduction in 2D

CFEM



$(M_{el})^{e_2}=[2\ 3\ 5\ 6]$

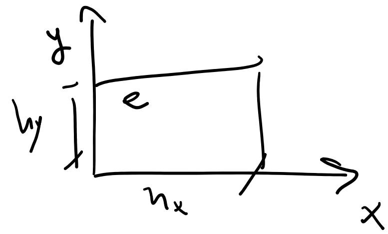


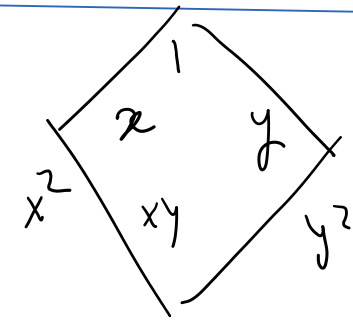
$$m^e_{4\times4} = \int N^T C_v T \, dv = \frac{C_v L_x L_y}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ & 4 & 2 & 1 \\ & \text{sym} & 4 & 2 \\ & & & 4 \end{bmatrix}$$

we need to assemble M^e for each element into global mass

e.g e^2: $M^{e_2} = \begin{array}{c} 2 \\ 3 \\ 5 \\ 6 \end{array}\left[\begin{array}{cccc} 2 & 3 & 5 & 6 \\ & & & \\ & & & \end{array}\right]$

DG:





$$T = \sum_{i=1}^{4} a_i \phi_i, \quad \phi_1 = 1, \ \phi_2 = x \quad \phi_3 = y \quad \phi_4 = xy \qquad N^e = [1 \quad x \quad y \quad xy]$$

$$M^e = \int_e N^{eT} C_v N^e \, dv$$

For DG $M^e = C_v L_x L_y \begin{bmatrix} 1 & \frac{L_x}{2} & \frac{L_y}{2} & \frac{L_x L_y}{2} \\ & \frac{L_x^2}{3} & \frac{L_x L_y}{2} & \frac{L_x L_y}{6} \\ \text{sym} & & \frac{L_y^3}{3} & L_x L_y^2/6 \\ & & & \frac{L_x^2 L_y^2}{9} \end{bmatrix}$

24 dof!

e_4 $a_{13}-a_{16}$	e_5 $a_{17}-a_{20}$	e_6 $a_{21}-a_{24}$
e_1 a_1-a_4	e_2 a_5-a_8	e_3 a_9-a_{12}

$$M_{24\times24} = \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_8 & e_6 \\ 1-4 & 5-8 & 9-12 & 13-16 & 17-20 & 21-24 \end{array}$$
$$\begin{bmatrix} M^{e_1} & & & & & \\ & M^{e_2} & & & & \\ & & M^{e_3} & & & \\ & & & M^{e_4} & & \\ & & & & M^{e_5} & \\ & & & & & M^{e_6} \end{bmatrix}$$

This is a block diagonal matrix:)

$$M^{-1} = \begin{bmatrix} M^{e_1^{-1}} & & \\ & & \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} M^{1^{-1}} & & & \\ & M^{2^{-1}} & & \\ & & \ddots & \\ & & & M^{8^{-1}} \end{bmatrix}$$

In DG methods for explicit schemes, we don't even form the mass matrix and only solve the problem at the element level:

$$m^e a^e = F^e \longrightarrow$$ contributions force source terms, natural, essential BC's and interaction with neighbor elements (ϕ)

We can solve each element's unknown at the element level if DG + an explicit solution scheme is used because the mass matrix is BLOCK DIAGONAL for DG method.

Discussion points:

- If an explicit method is used, only M appears on the LHS, so only the mass matrix determines the complexity of the solution scheme.
  - DG methods have a block diagonal mass matrix -> one element at a time solution scheme. This makes DG method much more efficient even though it has many more dofs.
  - For CFEMs, we have a sparse but not a block diagonal mass matrix. So, the system solve is more difficult.
    - One remedy is mass lumping