2022/09/27

Tuesday, September 27, 2022 9:57 AM

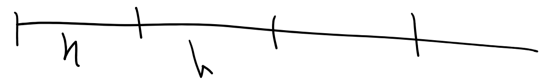
HW2 comment

$$C\dot{T} - k\,\Delta T = Q \qquad \nu = \frac{k}{C} \qquad [\nu] = \frac{L^2}{T}$$

1D $\qquad T_{,xx}$

For an explicit time marching scheme, time step depends on diffusion coefficient

$$\Delta t_{max} = f(\underset{\downarrow\, L}{h}, \underset{\downarrow\, L^2/T}{\nu}) \propto \frac{h^2}{\nu}$$



Time step of parabolic equations scale as square of the element size.

$$\Delta t_{max} = \min \left(C(p_h) \frac{h^{e^2}}{2\nu^e} \right)$$

correction factor based on element order

For HW2, you'll numerically investigate the stability limit for p = 1, 1D parabolic equation.

Hyperbolic $\frac{h}{C}$ — wave speed

Section 2

- Connection of DG methods and Interior Penalty (IP) methods
- The effect of target(star) values of DG formulation

$\begin{pmatrix}\hat{T} \\ \hat{q}\end{pmatrix}$ = weight of T, $q = -k\nabla T$ (1F formulation)

Starting from the WRS for the parabolic heat conduction:

$$R_e(\hat{T}, T; q, \hat{q}) = \int_e \hat{T}\,\underbrace{(\nabla\cdot q - Q)}_{R_i}\,dv + \int_{\partial e} \hat{T}\,\underbrace{(q^*_n - q\cdot n)}_{R_f}\,ds + \varepsilon \int_{\partial e} \hat{q}\cdot n\,\underbrace{(T^* - T)}_{R_u}\,ds$$

$(q, \hat{q})$ can be independently interpolated

$$\varepsilon : \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}$$

This is a number that we choose, and each option results in a slightly different formulation

We apply the divergence theorem to obtain:

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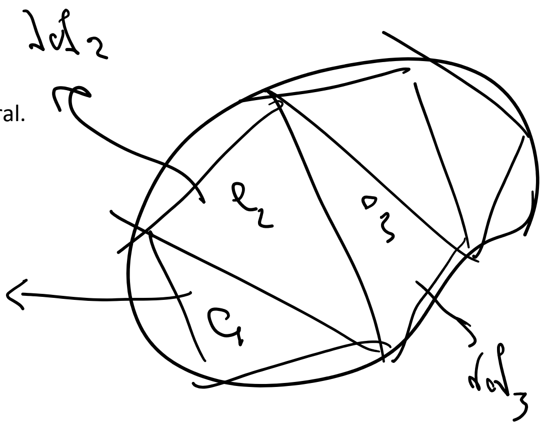
$$\boxed{R(\hat{T},T) = \underbrace{\int_e (\nabla \hat{T} q - \hat{T} Q)\,dV}_{\text{interior}} + \underbrace{\int_{\partial e} \hat{T} q_n^* \,ds + \varepsilon \int_{\partial e} \hat{q}\cdot n\,(T^* - T)\,ds}_{\text{boundary}} = 0}$$

Objectives:

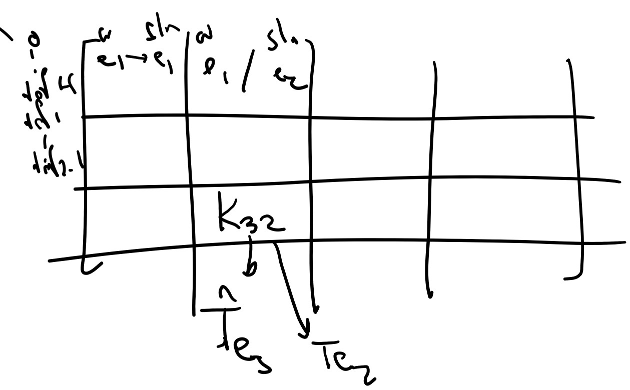
- The effect of ε on the form of stiffness matrix.
- Formulas for stiffness and residual coming from different face types and interior integral.
- Relation to IP methods.



parabolic

$$M\dot{a} + \underbrace{K}_{n\times n} \underbrace{a}_{\text{solutions}} = f$$

K weights



This is a linear problem and we can form the bilinear form B and the right hand side force term L

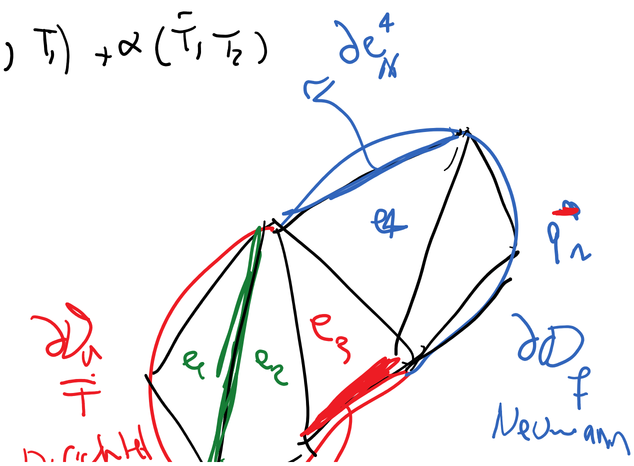
$$R(\hat{T},T) = \underbrace{B(\hat{T},T)}_{\text{bilinear form}} - L(\hat{T}) \longrightarrow \text{does not depend on } T$$

always true $\longleftarrow B(\hat{T}_1 + \alpha \hat{T}_2, T) = B(\hat{T}_1, T) + \alpha B(\hat{T}_2, T)$

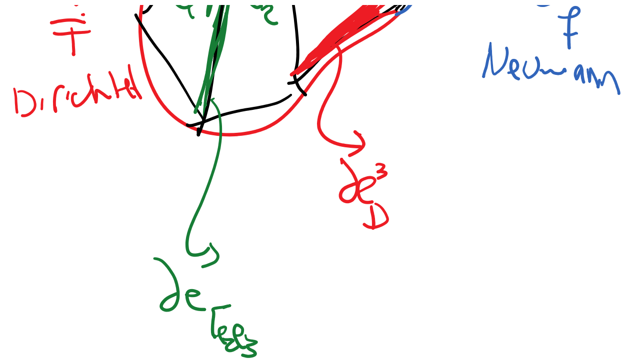
linear problems $\longleftarrow B(\hat{T}, T_1 + \alpha T_2) = B(\hat{T}, T_1) + \alpha(\hat{T}_1, T_2)$

$$\underbrace{K}_{B}\; a = \underbrace{f}_{L}$$

$\Gamma = \{$ all interior boundaries $\}$

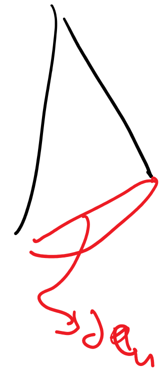


We want to find the contribution of Dirichlet, Neumann, interior facets and the interior of the element to B and L



Essential BC: $T^* = \bar T \qquad q_n^* = q\cdot n$

$$\underset{\text{essential BC}}{R_u(\hat T, T)} = \int_{\partial e_u} \hat T q_n^* \, ds + \varepsilon \int_{\partial e_u} \hat q \cdot n \,(\overset{\bar T}{T^*} - T)\, ds$$



$$R_u(\hat T, T) = \int_{\partial e_u} \left(\hat T q\cdot n - \varepsilon \hat q \cdot n T\right) ds + \varepsilon \int_{\partial e_u} \hat q \cdot n \bar T \, ds$$

$$= B_u(\hat T, T) - \underbrace{L_u(\hat T)}_{\text{Dirichlet}}$$

Essential BC

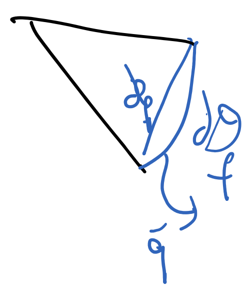
$$B_u(\hat T, T) = \int_{\partial e_u} \left(\hat T q \cdot n - \varepsilon \hat q \cdot n T\right) ds$$

$$L_u(\hat T) = -\varepsilon \int_{\partial e_u} \hat q \cdot n \bar T \, ds$$

(1)

2. Natural BC

$$R_f(\hat T, T) = \int_{\partial e_f} \hat T \underset{\bar q}{q_n^*} \, ds + \varepsilon \int_{\partial e_f} \hat q \cdot n \,(\underset{T^* = T}{T^*} - T)\, ds$$



$$R_f(\hat T, T) = -L_f(\hat T)$$

$$L_f(\hat T) = -\int_{\partial e_f} \hat T \bar q \, ds$$

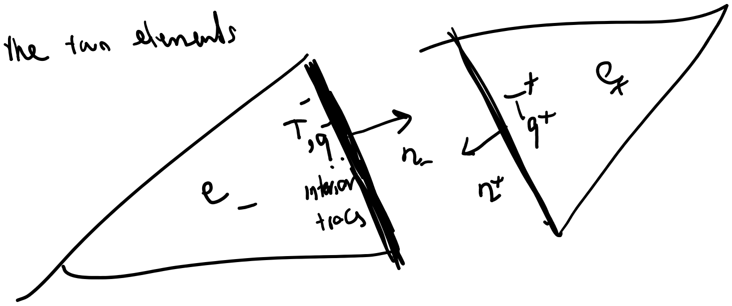
(2)

Note

node $\quad \partial_{ef} \quad R_{\partial f}(\hat{T}, T) = 0$

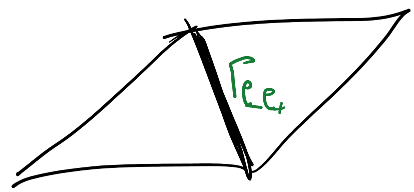
(3) Interior faces of the domain

$T^{\mp}, q^{\mp}$ are interior traces of T & q from the two elements



$$R^{e_-}_{\Gamma_{e^-e^+}}(\hat{T}^-, T^-) = \int_{\Gamma_{e^-e^+}} \hat{T}^- q^*_{n^-} + \varepsilon \int_{\Gamma_{e^-e^+}} \hat{q}^- \cdot n^- (T^* - T^-)\, ds$$

$\hat{T}^- \to \hat{q}^-$



$$R^{e_+}_{\Gamma_{e e^+}} = \int_{\Gamma_{e^-e^+}} \hat{T}^+ q^*_{n^+} + \varepsilon \int_{\Gamma_{e^-e^+}} \hat{q}^+ \cdot n^+ (T^* - T^+)\, ds$$

We add the terms together:

$$R_{\Gamma_{e^-e^+}}(\hat{T}, T) = \int_{\Gamma_{e^-e^+}} \left(\hat{T}^- q^*_{n^-} + \hat{T}^+ q^*_{n^+}\right) + \varepsilon \int_{\Gamma_{e^-e^+}} \hat{q}^- \cdot n^- (T^* - T^-) + \hat{q}^+ \cdot n^+ (T^* - T^+)\, ds$$

$$q^*_{n^-} = (q^*) \cdot n_- \qquad q^*_{n^+} = q^* \cdot n^+$$

$$\to \boxed{R_{\Gamma_{e^-e^+}} = \int_{\Gamma_{e^-e^+}} \left(\hat{T}^- q^* \cdot n^- + \hat{T}^+ q^* \cdot n^+\right) ds + \varepsilon \int_{\Gamma_{e^-e^+}} \hat{q}^- \cdot n^- (T^* - T^-) + \hat{q}^+ \cdot n^+ (T^* - T^+)\, ds}$$

Arnold_2000_Discontinuous Galerkin methods for elliptic problems

K_2, respectively. If v is a function on $K_1 \cup K_2$, but possibly discontinuous across e, let v_i denote $(v|_{K_i})|_e$, $i = 1, 2$. For a scalar function v we then define

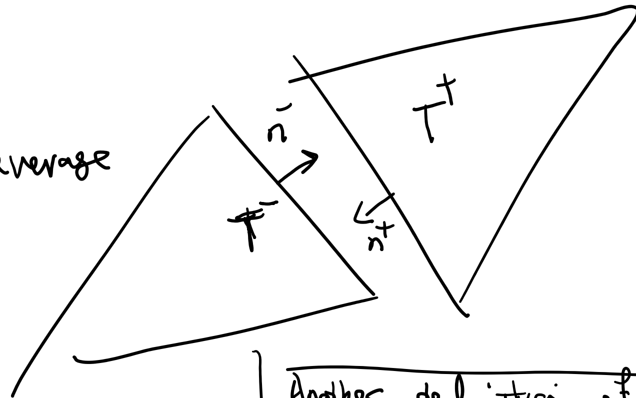
definition

$$\{v\} := \frac{1}{2}(v_1 + v_2), \qquad [\![v]\!] := v_1 n_1 + v_2 n_2.$$

If τ is a vector-valued function, we set

$$\{\tau\} := \frac{1}{2}(\tau_1 + \tau_2), \qquad [\![\tau]\!] := \tau_1 \cdot n_1 + \tau_2 \cdot n_2.$$

Notations of jump and averages for scalars and vectors



$$\underbrace{\{T\}}_{\text{scalar}} = \frac{1}{2}(T^- + T^+) \quad \text{scalar average}$$

Jump: There are various definitions of the jump

Arnold 2000, 2002, etc

$$\underbrace{[\![T]\!]}_{\text{scalar}} = \underbrace{T^- n^- + T^+ n^+}_{\text{vector}} = (T^+ - T^-)(-n^-)$$

sym. definition

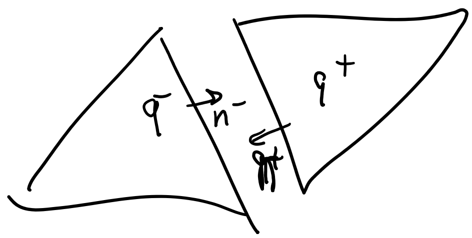
Another definition of jump not used here

$[f] = f_{out} - f_{in}$

from e^- side

$\{f\} = f^+ - f^-$

vector values



$$\{q\} = \frac{1}{2}(q^- + q^+)$$

$$[\![q]\!] = \underbrace{q^- \cdot n^- + q^+ \cdot n^+}_{\text{scalar}} = -(q^+ - q^-) \cdot n^-$$

Summary:

scalar T	$\{T\} = \frac{1}{2}(T^- + T^+)$	$[\![T]\!] = T^- n^- + T^+ n^+$	vector
vector q	$\{q\} = \frac{1}{2}(q^- + q^+)$	$[\![q]\!] = q^- \cdot n^- + q^+ \cdot n^+$	scalar

$$R_{e^-e^+} = \int_{\Gamma_{e^-e^+}} \underbrace{\left(\hat{T}^- q^* \cdot n^- + \hat{T}^+ q^* \cdot n^+\right)}_{\mathrm{I}} ds + \varepsilon \int_{\Gamma_{e^-e^+}} \underbrace{\hat{q}^- \cdot n^- (T^* - T^-) + \hat{q}^+ \cdot n^+ (T^* - T^+)}_{\mathrm{I}_2} ds$$

$$\int_{\Gamma_{e^-e^+}} \underbrace{\qquad\qquad}_{I_1} \qquad \int_{\Gamma_{e^-e^+}} \underbrace{\qquad\qquad}_{I_2}$$

$$I_1 = \hat{T}^- q^* \cdot n^- + \hat{T}^+ q^* \cdot n^+ = (\hat{T}^- n^- + \hat{T}^+ n^+) \cdot q^* = [[\hat{T}]] \cdot q$$

$$I_2 = \underbrace{\hat{q}^- \cdot n^- T^* + \hat{q}^+ \cdot n^+ T^*}_{(I_2)_a} - \underbrace{(\hat{q}^- \cdot n^- T^- + \hat{q}^+ \cdot n^+ T^+)}_{I_{2b}}$$

$$[[\hat{q}]] T^* \quad - I_{2b}$$

$$I_{2b} = \hat{q}^- \cdot n^- T^- + \hat{q}^+ \cdot n^+ T^+ = \hat{q}^- \cdot n^- \left(\frac{T^- + T^+}{2} + \frac{T^- - T^+}{2} \right) + \hat{q}^+ \cdot n^+ \left(\frac{T^+ + T^-}{2} + \left(\frac{T^+ - T^-}{2} \right) \right)$$

$$= (\hat{q}^- \cdot n^- + \hat{q}^+ \cdot n^+) \left(\frac{T^- + T^+}{2} \right) + \frac{\hat{q}^-}{2} \cdot (T^- n^- - T^+ n^-) + \frac{\hat{q}^+}{2} \cdot (n^+ T^+ - n^+ T^-)$$

$$+ T^+ n^+ \qquad + n^+ T^+$$

$$= [[\hat{q}]] \{T\} + \underbrace{\left(\frac{\hat{q}^- + \hat{q}^+}{2} \right)}_{\{\hat{q}\}} \underbrace{(T^- n^- + T^+ n^+)}_{[[T]]}$$

$$I_{2b} = [[\hat{q}]] \{T\} + \{\hat{q}\} [[T]]$$

we'll obtain

$$(3) \quad \boxed{R_{\Gamma_{e^-e^+}}(\hat{T}, T) = \varepsilon \int_{\Gamma_{e^-e^+}} [[\hat{q}]] (T^* - \{T\}) \, ds + \int_{\Gamma_{e^-e^+}} [[\hat{T}]] \cdot q^* \, ds}$$