2022/09/29

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From last time

For interior boundaries

$$R_{\Gamma_{e\bar{e}^+}}(\hat{T}, T) = \int_{\Gamma_{e\bar{e}^+}} I_1 \, ds + \varepsilon \int_{\Gamma_{e\bar{e}^+}} I_2 \, ds$$

$$= \int_{\Gamma_{e\bar{e}^+}} [[\hat{T}]] \cdot q^* \, ds + \varepsilon \int_{\Gamma_{e\bar{e}^+}} \left([[\hat{q}]] T^* - [[\hat{q}]] \{T\} - \{\hat{q}\} [[T]] \right) ds$$

$$\boxed{R_{\Gamma_{e\bar{e}^+}}(\hat{T}, T) = \varepsilon \int_{\Gamma_{e\bar{e}^+}} \left([[\hat{q}]] \left(T^* - \{T\} \right) - \{\hat{q}\} [[T]] \right) ds + \int_{\Gamma_{e\bar{e}^+}} [[\hat{T}]] \, q^* \, ds}$$

Interior terms

I

Next, we'll discuss the star value options:
The star values for elliptic and parabolic equations are generally the same.
Now, we are solving an elliptic PDE without the $C\dot{T}$

Read Arnold 2000 and 2002 papers

Local Discontinuous Galerkin (LDG) fluxes

Cockburn & Shu

$$\left\{ \begin{array}{l} q^* = \{q\} + \vec{\beta} [[q]] + \kappa \alpha [[T]] \\ T^* = \{T\} + \gamma_0 [[T]] \end{array} \right. \qquad (1)$$

conductivity

has a stabilization effect.

Previously, we used average fluxed for the parabolic heat conduction problem.
Here we are solving an elliptic PDE (C Tdot is absent from the PDE). So, we need to use elliptic fluxes.
Formula 1 is a general form of star values for elliptic PDEs

Method	$h_\sigma^{e,K}$ (σ^*)	$h_u^{e,K}$ (u^*)
Bassi–Rebay 1	$\{\sigma_h\}$	$\{u_h\}$
Brezzi et al. 1	$\{\sigma_h\} - \eta^e \{r_e([\![u_h]\!])\}$	$\{u_h\}$
LDG	$\{\sigma_h\} - \eta^e [\![u_h]\!] + \beta^e [\![\sigma_h]\!]$	$\{u_h\} + \gamma^e [\![u_h]\!]$
IP	$\{\nabla u_h\} - \eta^e [\![u_h]\!]$	$\{u_h\}$
Bassi–Rebay 2	$\{\nabla u_h\} - \eta^e \{r_e([\![u_h]\!])\}$	$\{u_h\}$
Baumann–Oden	$\{\nabla u_h\}$	$\{u_h\} - [\![u_h]\!] \cdot n_K$
Babuška-Zlámal	$-\eta^e [\![u_h]\!]$	$u_h\|_K$
Brezzi et al. 2	$-\eta^e \{r_e([\![u_h]\!])\}$	$u_h\|_K$

$T \to u$

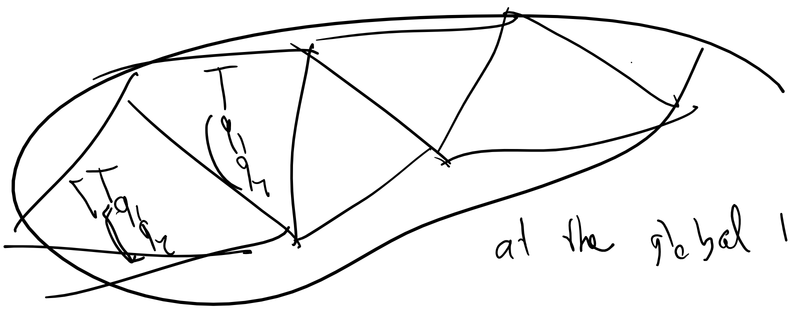
$\nabla u \to \sigma \quad (-\kappa\sigma = q)$

$\eta [[T]]$ is an operator on $[[T]]$
a simple choice is like above
$\eta([[T]]) = \alpha [[T]]$

why not have a term like $[[q]]$ in $\hat{T}$

$$\hat{T} = \{T\} + \gamma[[T]] - C[[q]]$$

If we don't have this term (which is the case) through a lengthy process for 2-Field formulations where T & q are both primary solution unknowns q can be condensed out from the global system

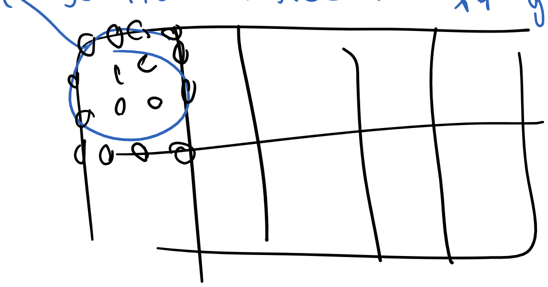


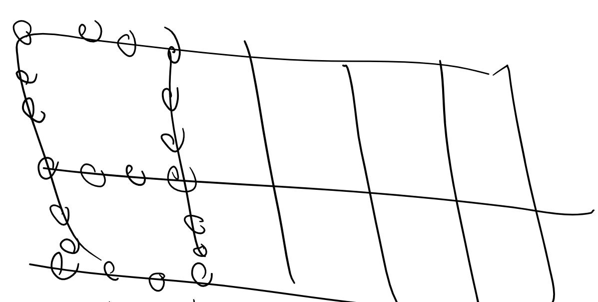
$$\hat{T} = \{T\} + \gamma\cdot[[T]] - \cancel{C[[q]]}$$

at the global level we only deal with unknowns of T ($\frac{1}{3}$ of all dofs in 2D if T & q have the same polynomial order) $\longrightarrow$ we go back to elements after T is solved & locally solve for each element's q.

Similar to CFEM static condensation

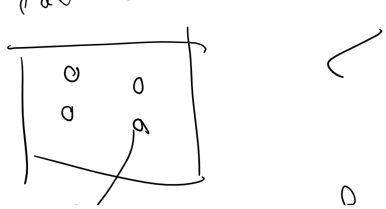
get rid of inside dof for global solve





Solve this

each element



find these dets

Flux equation: The effect of each term

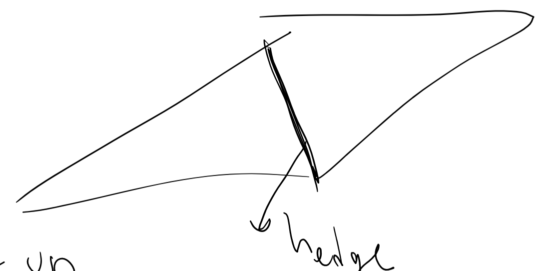
$$q^* = \{q\} + \vec{\beta}\,[[q]] + k\,\underbrace{\alpha}_{\frac{\eta_0}{\tilde{h}}}\,[[T]]$$

$[\rho]$ = physical dimension

$$[\alpha] = \frac{[q]}{[k][T]} = \frac{[k][T]/L}{[k][T]} = \frac{1}{L} \qquad q = -k\nabla T$$

α has the dimension of $\frac{1}{\text{length}}$

$$\boxed{\alpha = \frac{\eta_0}{\tilde{h}}}$$



there are so many choices to come up with a length scale from the two neighboring elements. Easiest one is $\tilde{h} = \frac{1}{h_{edge}}$

What should be chosen for η_0 for Elliptic PDEs?

TABLE 6.1
Properties of the DG methods

condition for stability

Method	cons.	a.c.	stab.	type	cond.	H^1	L^2
Brezzi et al. [18]	✓	✓	✓	α^r	$\eta_0 > 0$	h^p	h^{p+1}
LDG [35]	✓	✓	✓	α^j	$\eta_0 > 0$	h^p	h^{p+1}
IP [43]	✓	✓	✓	α^j	$\eta_0 > \eta^*$	h^p	h^{p+1}
Bassi et al. [10]	✓	✓	✓	α^r	$\eta_0 > 3$	h^p	h^{p+1}
NIPG [53]	✓	×	✓	α^j	$\eta_0 > 0$	h^p	h^p
Babuška–Zlámal [6]	×	×	✓	α^j	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Brezzi et al. [19]	×	×	✓	α^r	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Baumann–Oden ($p = 1$)	✓	×	×	-	-	×	×
Baumann–Oden ($p \geq 2$)	✓	×	×	-	-	h^p	h^p
Bassi–Rebay [9]	✓	✓	×	-	-	$[h^p]$	$[h^{p+1}]$

interior penalty

only T is interpolated

*'s from (1)

$\eta_0 > 0$

T & q are interpolated and q can be condensed out from the global system

— $\vec{\gamma}$ & $\vec{\beta}$ are two user defined vectors.

— They can change point-to-point

— Generally $\vec{\gamma} = -\vec{\beta}$

As we'll see one choice of these values results in alternating fluxes.

For now, I choose these values to be zero

(2)
$$\boxed{\begin{aligned} q^{*} &= \{q\} + \bar{\alpha}[[T]] \qquad \bar{\alpha} = \alpha k, \ \alpha = \frac{\eta_0}{h} \\ T^{*} &= \{T\} \end{aligned}}$$

Recall
$$R_{\Gamma_{e^-e^+}}(\hat{T},T) = B_{\Gamma_{e^-e^+}}(\hat{T},T) + \varepsilon \int_{\Gamma_{e^-e^+}} \left([[\hat{q}]](T^{*} - \{T\}) - \{\hat{q}\}[[T]] \right) ds + \int_{\Gamma_{e^-e^+}} [[\hat{T}]] q^{*} ds$$

from eq (2)

for average + $\bar{\alpha} > 0$ option we have:

$$B_{\Gamma_{e^-e^+}}(\hat{T},T) = \int_{\Gamma_{e^-e^+}} [[\hat{T}]](\{q\} + \bar{\alpha}[[T]]) ds + \varepsilon \int_{\Gamma_{e^-e^+}} -\{\hat{q}\}[[T]] \, ds$$

$$\boxed{B_{\Gamma_{e^-e^+}}(\hat{T},T) = \int_{\Gamma_{e^-e^+}} [[\hat{T}]] \bar{\alpha} [[T]] ds + \int_{\Gamma_{e^-e^+}} \left([[\hat{T}]]\{q\} - \varepsilon\{\hat{q}\}[[T]]\right) ds}$$

(3) interior interface bilinear form for average + $\bar{\alpha} > 0$ choice (2)

Interior term

2 field formulation

$$R_i(\hat{T},T) = \int_e (-\nabla\hat{T} q - \hat{T} Q) ds + \int \hat{q}\,(q + k\nabla T)\, dv$$

T & q are interpolated

let's just continue with q field

let's just continue with q field

$$R_i(\hat{T}, T) = \int_e \nabla\hat{T}\, \underbrace{\kappa \nabla T}_{-q}\, dV - \int_e \hat{T} Q\, ds$$

$\underbrace{\int_e \nabla\hat{T}\,\kappa\nabla T\,dV}_{\text{goes to the bilinear for ...}}$ $\underbrace{\int_e \hat{T} Q\,ds}_{\text{goes to the RHS}}$

T & q are interpolated for LDG

for T field (e.g. IP) we only have T and this term is dropped

$$\boxed{\begin{aligned} B_i^e(\hat{T}, T) &= \int_e \nabla\hat{T}\,\kappa\nabla T\,dV \quad + \text{ for 2 field} \\ L_i^e(\hat{T}) &= \int_e \hat{T} Q\, dV \end{aligned}}$$

(4)

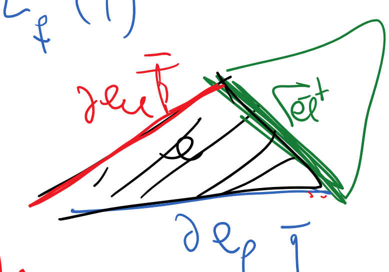
$$B(\hat{T}, T) = L(\hat{T})$$

$$B(\hat{T}, T) = \sum_e B_i^e(\hat{T}, T) + \sum_{\partial e_u} B_u^e(\hat{T}, T) + \sum_{\Gamma_{ee'}} B_{\Gamma_{ee'}}(\hat{T}, T)$$

$\Gamma_{ee'}$ ← interior interfaces

$$L(\hat{T}) = \sum_e L_i^e(\hat{T}) + \sum_{\partial e_u} L_u^e(\hat{T}) + \sum_{\partial e_f} L_f^e(\hat{T})$$

$$B_i^e = \int -\nabla\hat{T}\,\kappa\nabla T\, ds \qquad L_i^e = \int_e \hat{T} Q\, ds$$



$$B_u^e = \int_{\partial e_u} \left(\hat{T} q - \varepsilon T \hat{q}\right) ds \qquad L_u^e(\hat{T}) = -\varepsilon \int \hat{q}\cdot n\, \bar{T}\, ds$$

$$B_f^e = 0 \qquad L_f^e(\hat{T}) = \int_{\partial e_f} \hat{T}\bar{q}\, ds$$

$$B_{\Gamma_{ee'}} = \int [\hat{T}]\left(\{q\} + \tilde{\alpha}[T]\right) ds - \varepsilon \int \{\hat{q}\}[T]\, ds \quad \underline{\text{eq 3}}$$

(5)

Bilinear forms from Arnold 2002

TABLE 4.1
Bilinear forms restricted to $V_h \times V_h$ for some DG methods.

Method	$B_h(u,v)$
Bassi–Rebay [9]	$(\nabla_h u + R(u), \nabla_h v + R(v))$
Brezzi et al. [18]	$(\nabla_h u + R(u), \nabla_h v + R(v)) + \alpha^{\mathsf{r}}(u,v)$
LDG [35]	$(\nabla_h u + R(u) + L_\beta(u), \nabla_h v + R(v) + L_\beta(v)) + \alpha^{\mathsf{j}}(u,v)$
IP [43]	$(\nabla_h u, \nabla_h v) + (R(u), \nabla_h v) + (\nabla_h u, R(v)) + \alpha^{\mathsf{j}}(u,v)$
Bassi et al. [10]	$(\nabla_h u, \nabla_h v) + (R(u), \nabla_h v) + (\nabla_h u, R(v)) + \alpha^{\mathsf{r}}(u,v)$
Baumann–Oden [12]	$(\nabla_h u, \nabla_h v) - (R(u), \nabla_h v) + (\nabla_h u, R(v))$
NIPG [53]	$(\nabla_h u, \nabla_h v) - (R(u), \nabla_h v) + (\nabla_h u, R(v)) + \alpha^{\mathsf{j}}(u,v)$
Babuška–Zlámal [6]	$(\nabla_h u, \nabla_h v) + \alpha^{\mathsf{j}}(u,v)$
Brezzi et al. [19]	$(\nabla_h u, \nabla_h v) + \alpha^{\mathsf{r}}(u,v)$