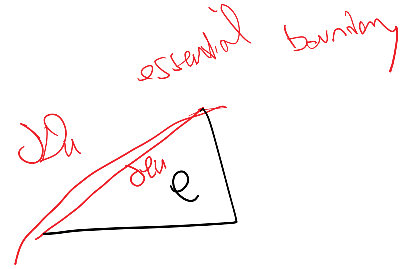
Today: components of stiffness matrix and investigating its symmetry

a. Essential BC




$$B_u(\hat{T}, T) = \int_{\partial e_u} (\hat{T} q - \varepsilon T \hat{q}) \, n \, ds$$

$$T = [T_1(x) \; \dots \; T_n(x)] \begin{bmatrix} a_1(t) \\ \vdots \\ a_n(t) \end{bmatrix} \longrightarrow q = -k\nabla T \;\; \text{field}$$

For parabolic $a$'s depend on time

For Elliptic considered, $a$'s are constant

$q$: directly interpolated $= [q_1(x) \; \dots \; q_n(x)]_{2 \times n} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

this has $x$ & $y$ component in 2D

$K_{ij}$ from $B_u(\hat{T}, T)$

$$K_{ij} = B_u(T_i, T_j)$$

$T_i$: weight funct.

solution $T_j$

weight $T_i$

$\hat{T} = T_i \qquad T = T_j$

$$K_{ij} = \int_{\partial e_u} (T_i \, q_j - \varepsilon T_j \, q_i) \, n \, ds$$

$q_j$: basis $\#j$ for $q$

1F: $q_i = -k\nabla T_i$

2F: it's direct basis for $q$'s

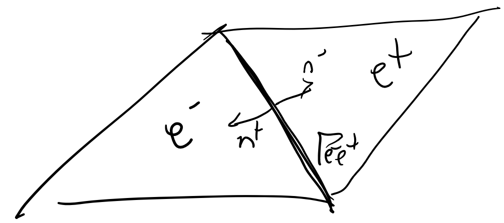
$$\boxed{\begin{array}{ll} K_{ji} = \int_{\partial e_u} (T_j \, q_i - \varepsilon T_i \, q_j) \, n \, ds & \\ i \to j, \; j \to i & \\ \varepsilon = -1 & K_{ji} = K_{ij} \;\; \text{sym} \\ \varepsilon = 1 & K_{ji} = -K_{ij} \\ \varepsilon = 0 & \text{neither one} \end{array}}$$

① Essential BC $K$

$\varepsilon = -1$ is the option that I provided a sample DG formulation early in the course

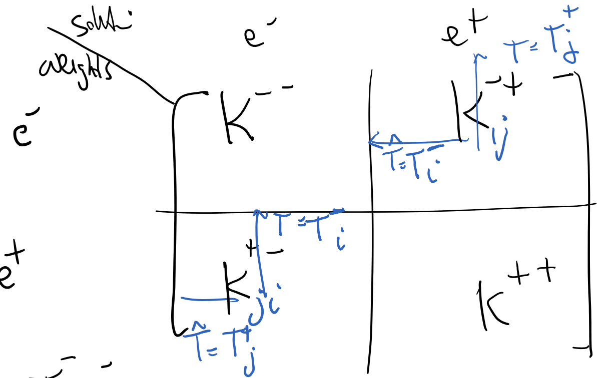
b. Interior faces

$$B_{\Gamma_{e^-e^+}}(\hat{T},T)=\int_{\Gamma_{e^-e^+}}[\hat{T}]\left(\{q\}+\bar{\alpha}[T]\right)ds-\varepsilon\int_{\Gamma_{e^-e^+}}\{\tilde{q}\}[T]\,ds$$




bI Off-diagonal blocks

$$K_{ij}^{-+}=B_{\Gamma}(\hat{T},T)\quad \text{for } \hat{T}=T_i^-,\ T=T_j$$




$$[\hat{T}]=\hat{T}^-n^-+\hat{T}^+n^+=T_i\,n^-+0\,n^+=T_i\,n^-$$

$$[T]=T^-n^-+T^+n^+=\quad=T_j^+\,n^+$$

$$\{\hat{q}\}=\frac{\hat{q}^-+\hat{q}^+}{2}=\frac{q_i+0}{2}=q_i/2\qquad \{q\}=\frac{q^-+q^+}{2}=\frac{q_j}{2}$$

$$K_{ij}^{-+}=\int_{\Gamma_{e^-e^+}}\left\{\underbrace{(T_i^-n^-)}_{[\hat{T}]}\left(\underbrace{\frac{q_j^+}{2}}_{\{q\}}+\bar{\alpha}\underbrace{(T_j^+n^+)}_{[T]}\right)-\varepsilon\underbrace{\left(\frac{q_i}{2}\right)}_{\{\hat{q}\}}\underbrace{(T_j^+n_j^+)}_{[T]}\right\}ds$$

$$K_{ij}^{-+}=\int_{\Gamma_{e^-e^+}}\left(\bar{\alpha}\,T_i^-T_j^+\underbrace{n^-n^+}_{-1}+\frac{n^-}{2}\left(T_i^-q_j^++\varepsilon\,T_j^+q_i^-\right)\right)ds$$

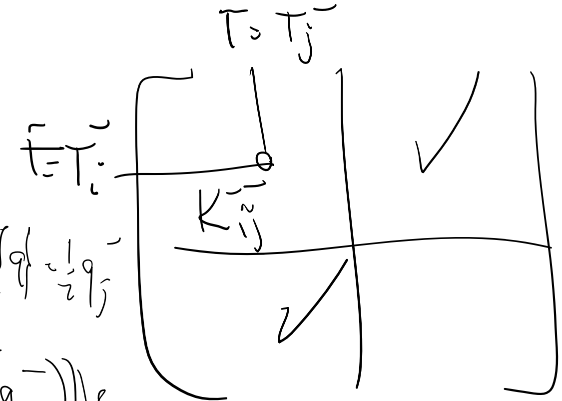
$$K_{ij}^{-+}=\int_{\Gamma_{e^-e^+}}\left(\underbrace{\boxed{-\bar{\alpha}\,T_i^-T_j^+}}_{a}+\frac{n^-}{2}\left(\underbrace{\boxed{T_i^-q_j^+}}_{b}+\underbrace{\boxed{\varepsilon\,T_j^+q_i^-}}_{c}\right)\right)ds$$

$K_{ji}^{+-}$ $\quad i\to j$, $j\to i$, $+\to-$, $-\to+$

$$K_{ji}^{+-}=\int_{\Gamma_{e^-e^+}}\underbrace{\bar{\alpha}\,T_j^+T_i^-}_{a}+\cancel{\frac{n^+}{2}}\ +\frac{n^-}{2}\left[\underbrace{-\varepsilon\,T_i\,q_j^+}_{\hookrightarrow b}\underbrace{-T_j^+\,q_i^-}_{\hookrightarrow c}\right]ds$$

$$\begin{cases}\varepsilon=-1 & \text{sym}\\ \varepsilon=1\ \&\ \alpha=0 & \text{skew}\\ \text{else} & \text{neither}\end{cases}$$

$$B_{\Gamma_{ee^+}}(\hat T, T) = \int_{\Gamma_{ee^+}} [\hat T]\left(\{q\} + \bar\alpha [T]\right) ds - \varepsilon \int_{\Gamma_{ee^+}} \{\tilde q\}[T]\, ds$$




$$[\hat T] = T_i^- n^- \quad \{\hat q\} = \frac{1}{2} q_i^- \quad [T] = T_j^- n^-, \quad \{q\} = \frac{1}{2} q_j^-$$

$$K_{ij}^{--} = \int \left(\bar\alpha T_i^- T_j^- + \frac{n^-}{2}\left(T_i^- q_j^- - \varepsilon T_j^- q_i^-\right)\right) ds$$

$i \to j$
$j \to i$

$$K_{ji}^{--} = \int \left(\bar\alpha T_j^- T_i^- + \frac{n^-}{2}\left(-\varepsilon T_i^- q_j^- + T_j^- q_i^-\right)\right) ds$$

for
- $\varepsilon = -1$ — fully symmetric
- $\varepsilon = 1$ & $\bar\alpha = 0$ — skew sym.
- else — neither

$$K_{ij}^{-+} = \int_{\Gamma_{ee^+}} \left(\bar\alpha T_i^- T_j^+ \underbrace{n^- \cdot n^+}_{-1} + \frac{n^-}{2}\left(T_i^- q_j^+ + \varepsilon T_j^+ q_i^-\right)\right) ds$$

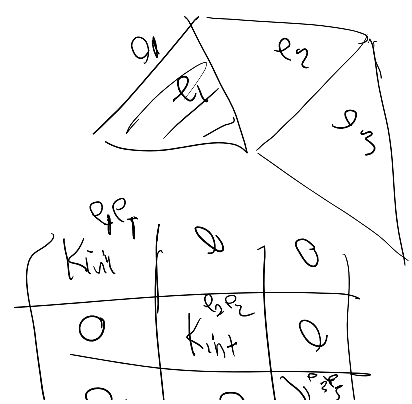

$$K_{ji}^{--} = \int \left(\bar\alpha T_j^- T_i^- + \frac{n^-}{2}\left(-\varepsilon T_i^- q_j^- + T_j^- q_i^-\right)\right) ds$$

for
- $\varepsilon = -1$ — fully symmetric
- $\varepsilon = 1$ & $\bar\alpha = 0$ — skew sym.
- else — neither


(2) Interior interface

c. Natural boundary -> no stiffness term here
d. Interior of the element

$$B^e_{int}(\hat T, T) = \int_e \nabla \hat T \kappa \nabla T \, dV$$




$$\boxed{K^e_{ij} = \int \nabla T_i^e \kappa \nabla T_j^e \, dV}$$

sym

(3)

sym

From ① to ③

$$\begin{cases} \varepsilon = -1 & \text{sym } K \\ \varepsilon = 1 \ \& \ \bar{\alpha} = 0 & \text{skew } K \text{ except interior element contribution} \\ \text{else} & \text{neither} \end{cases}$$

Discussion of the weak statement and comparison with CFEM

$$\sum_e \boxed{\int_e (\nabla \hat{T} \, k \nabla T) \, dV} \rightarrow B_i^o$$

$$B(\hat{T}, T) = L(\hat{T})$$

LHS     RHS

$$+ \sum_e \int_{\partial e_n} (T \hat{q} - \varepsilon \hat{T} q) \, ds$$

$$+ \sum_{\Gamma_{e e^+}} \int \Big( \underbrace{[\hat{T}]\{q\}}_{a} - \underbrace{\varepsilon \{\hat{q}\}[T]}_{b} + \underbrace{\bar{\alpha}[\hat{T}][T]}_{c} \Big) \, ds$$

$$= \boxed{\sum_e \int_e \hat{T} Q \, dV} + \sum_e \varepsilon \int_{\partial e_u} \hat{q} \cdot n \bar{T} \, ds - \boxed{\sum_e \int_{\partial e_f} \hat{T} \bar{q}_n \, ds}$$

$\partial e_n$

$e^+$   $e^-$

$\partial e_f$

▭ also in CFEM    weakly enforcing the essential BC in DG

Interior Penalty (IP) terms

Comparison with Riviere

1D elliptic PDE

$$\forall x \in (0,1), \quad -(K(x)p'(x))' = f(x), \tag{1.1}$$

$$p(0) = 1, \tag{1.2}$$

$$p(1) = 0, \tag{1.3}$$

$k$

$p$ solution (our $T$)

$v$ weight function

$$\sum_{n=0}^{N-1}\int_{x_n}^{x_{n+1}} \underbrace{K(x)p'(x)v'(x)}_{\nabla T\, K \nabla T}dx - \sum_{n=0}^{N}\underbrace{\{K(x_n)p'(x_n)\}[v(x_n)]}_{-\{q\}[\hat{T}]} + \epsilon\sum_{n=0}^{N}\underbrace{\{K(x_n)v'(x_n)\}}_{-\{\hat{q}\}}\underbrace{[p(x_n)]}_{[T]}$$

$$= \int_0^1 f(x)v(x)dx - \epsilon K(x_0)v'(x_0)p(x_0) + \epsilon K(x_N)v'(x_N)p(x_N)$$

$$= \int_0^1 \underbrace{f(x)v(x)dx}_{\text{solution}} - \epsilon K(x_0)v'(x_0).$$

We can add two penalty terms to the LHS ~~ferred to as penalty terms.~~

$$J_0(v,w) = \sum_{n=0}^{N}\frac{\sigma^0}{h_{n-1,n}}[v(x_n)][w(x_n)],$$

$$\frac{\sigma^0}{h}[\hat{T}][T]$$

$\sigma^0 = K\eta_0$

And

$$J_1(v,w) = \sum_{n=1}^{N-1}\frac{\sigma^1}{h_{n-1,n}}[v'(x_n)][w'(x_n)],$$

$$\frac{\sigma^1}{h}[\hat{T}'][T']$$

multi dimension

$$(\quad)[\hat{q}][q]$$

LDG flux

$$q^* = \{q\} + \vec{\beta}[q] + K\alpha[T], \quad \alpha = \frac{\eta_0}{h_{edge}}$$

$$T^* = \{T\} + \vec{\gamma}\cdot[T] + \underbrace{\delta_0\sigma^1[q]}_{}$$

$\sigma^1$ term would have created this

In LDG flux (by Cockburn and Shu) they didn't add the blue term so that element level q could be eliminated at the global level

-> Solve for T at global level

-> Go back to elements and solve for q

$\sigma^1 \neq 0$ not very common with IP methods

Variations of IP methods

(DG $\eta_0 \geq \eta^*$)

- If $\epsilon = -1$, $\sigma^1 = 0$, and $\sigma^0$ is bounded below by a large enough constant, the resulting method is called the symmetric interior penalty Galerkin (SIPG) method, introduced in the late 1970s by Wheeler [109] and Arnold [1].

I field

Formula

Arnold 2002

TABLE 6.1

Arnold 2002

Formulation

T interpolation

TABLE 6.1
*Properties of the DG methods*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| IP [43] | ✓ | ✓ | ✓ | $\alpha^j$ | $\eta_0 > \eta^*$ | $h^p$ | $h^{p+1}$ |

- If $\epsilon = -1$ and $\sigma^0 = \sigma^1 = 0$, the resulting method is called the global element method, introduced in 1979 by Delves and Hall [43]. However, the matrix associated with the bilinear form is indefinite, as the real parts of the eigenvalues are not all positive and thus the method is not stable.

not pos def nor neg def.

$\sigma^0 > 0$

- If $\epsilon = +1$, $\sigma^1 = 0$, and $\sigma^0 = 1$, the resulting method is called the nonsymmetric interior penalty Galerkin (NIPG) method, introduced in 1999 by Rivière, Wheeler, and Girault [95].

1F

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| NIPG [53] | ✓ | × | ✓ | $\alpha^j$ | $\eta_0 > 0$ | $h^p$ | $h^p$ |

- If $\epsilon = +1$ and $\sigma^0 = \sigma^1 = 0$, the resulting method was introduced by Oden, Babuška, and Baumann in 1998 [84]. Throughout these notes, we will refer to this method as the NIPG 0 method, since it corresponds to the particular case of NIPG with $\sigma^0 = 0$.

| Method | cons. | a.c. | stab. | type | cond. | $H^1$ | $L^2$ |
|---|---|---|---|---|---|---|---|
| Baumann–Oden ($p \geq 2$) | ✓ | × | × | - | - | $h^p$ | $h^p$ |

Not stable