2022/10/11

Tuesday, October 11, 2022 9:40 AM

Continuing from last time
Last time we discussed epsilon = -1,1
What about epsilon = 0

$$+\varepsilon \int \hat{q}\, n\, (T^{*} - T)\, ds$$

if $\varepsilon = 0$

we cannot enforce essential BC $\partial\Omega_u$

Unless q^* has the penalty term

$$q^* = \{q\}\ q\vec{\beta}[q]\ \underbrace{+\tilde{\alpha}[T]}_{K\alpha,\ \alpha = \frac{\eta_0}{h_e}}$$

on essential Boundary

$\bar{T}(-n) + T(n) = (T - \bar{T})\, n$

So having nonzero η_0 $\{\tilde{\alpha} = K\alpha,\ \alpha = \frac{\eta_0}{h_e}\}$ or in IP jargon σ_0 allows us to enforce essential BC.

- If $\epsilon = 0$, we obtain the incomplete interior penalty Galerkin (IIPG) method introduced by Dawson, Sun, and Wheeler [42] in 2004.

$\sigma_0 > 0$

All these previous formulations have 1 primary field interpolated (T).

Local DG (LDG) interpolates both T and q and has better stability properties

Arnold 2002

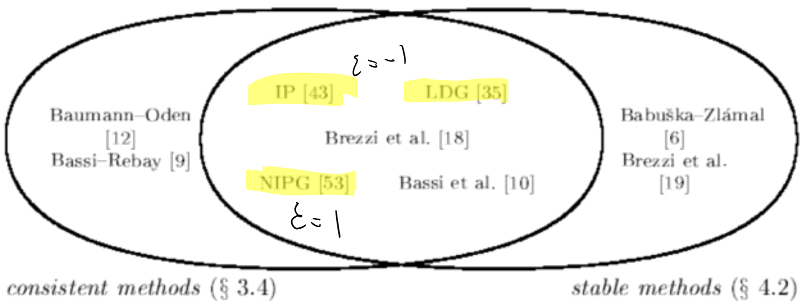
TABLE 6.1
Properties of the DG methods

	Method	cons.	a.c.	stab.	type	cond.	H^1	L^2	
	Brezzi et al. [18]	✓	✓	✓	α^r	$\eta_0 > 0$	h^p	h^{p+1}	→ 2-field (T,q)
$q = \alpha$	LDG [35]	✓	✓	✓	α^j	$\eta_0 > 0$	h^p	h^{p+1}	
	IP [43]	✓	✓	✓	α^j	$\eta_0 > \eta^*$	h^p	h^{p+1}	single field (T)
	Bassi et al. [10]	✓	✓	✓	α^r	$\eta_0 > 3$	h^p	h^{p+1}	
$\varepsilon = 1$	NIPG [53]	✓	×	✓	α^j	$\eta_0 > 0$	h^p	h^p	
	Babuška–Zlámal [6]	×	×	✓	α^j	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}	
	Brezzi et al. [19]	×	×	✓	α^r	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}	

$\varepsilon=1$ NIPG [53]	✓	×	✓	α^j	$\eta_0 > 0$	h^p	h^p
Babuška–Zlámal [6]	×	×	✓	α^j	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Brezzi et al. [19]	×	×	✓	α^r	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
$\varepsilon=1$ Baumann–Oden ($p=1$)	✓	×	×	-	- unstable	×	×
Baumann–Oden ($p \geq 2$)	✓	×	×	-	-	h^p	h^p
Bassi–Rebay [9]	✓	✓	×	-	-	$[h^p]$	$[h^{p+1}]$

$\eta_0 = 0$

FIG. 3.1. *Consistency and stability of some DG methods.*



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elliptic

$\varepsilon=-1$	SIPG	needs large η_0		1F
	LDG	needs	$\eta_0 > 0$	2F
$\varepsilon=1$	NIPG		$\eta_0 > 0$	1F stable
	oden-Babuska		$\eta_0 = 0$	1F unstable

for parabolic PDEs η_0 can be zero

Discussion of coercivity of the bilinear form

$$B(\hat{T}, T) = L(\hat{T})$$

$$\sum_e \int \nabla \hat{T} \, k \, \nabla T \, dv + \sum_e \int_{\partial e_h} (\tilde{T} q - \varepsilon T \hat{q}) \, n \, ds +$$

$$\sum \int_{e e^+} \left([\hat{T}] \{q\} - \varepsilon [T] \{\hat{q}\} \right) + \bar{\alpha} [\hat{T}][T] \, ds = \sum_e L_i^e(T) + \sum_{\partial e_h} L_h^e(\hat{T}) + \sum_{\partial e_f} L_f^e(\hat{T})$$

choose $\hat{T} = T$ i.e. the bilinear form choose $T=1$ in the bilinear form

$$\boxed{\begin{aligned} B(T,T) &= \sum_e \int_e \nabla T \, k \, \nabla T \, dv + \sum_e \int_{\partial e_n} (Tq - \varepsilon T q) \, n \, ds \\ &+ \sum_{\Gamma_{e\bar{e}+}} \int_{\Gamma_{e\bar{e}+}} \Big(\big([T]\{q\} - \varepsilon [T]\{q\} \big) + \bar{\alpha} [T][T] \Big) ds \end{aligned}}$$

$\star$

$\varepsilon = 1$ (NIPG)

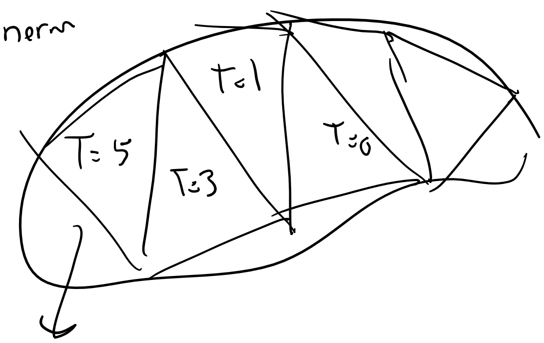
$$\boxed{B(T,T) = \underbrace{\sum_e \int \nabla T \, \underset{\text{pos def}}{k} \, \nabla T \, dv}_{\geq 0 \ \text{term 1}} + \underbrace{\sum_{\Gamma_{e\bar{e}+}} \int_{\Gamma_{e\bar{e}+}} \bar{\alpha}_0 [T][T] \, ds}_{\geq 0 \ \text{term 2}} \qquad \varepsilon = 1}$$

$B(T,T) \geq 0$

consider $\underline{\bar{\alpha} = 0}$ $\qquad T = c \rightarrow$ constant $\qquad B(T,T) = 0$

$T \neq 0$ but $B(T,T) = 0$

we have a seminorm



Side note
seminorm $\| \ \|$
1) $\|v\| \geq 0$
2) $\|v + w\| \leq \|v\| + \|w\|$
3) $\|\lambda v\| = |\lambda| \, \|v\|$

norm = seminorm + pos. def prop.
$\|v\| = 0 \iff v = 0$

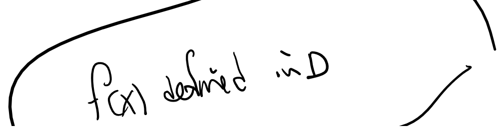
$\bar{\alpha} > 0 \qquad B(T,T) > 0$

$\bar{\alpha} > 0$ (second term) makes $B(T,T)$ a norm

In fact on can show that $B(T,T)$ is coercive

Definition of coercivity

$$\exists \lambda \ni \forall T \qquad B(T,T) \geq \lambda \underbrace{\|T\|^2}_{\text{a norm of } T \text{ in continuum}}$$



$$L_2(f) = \sqrt{\int_D |f|^2 \, dV}$$

$W_1(f)$ ...

$$H_1(f) = \sqrt{\int_D |f|^2 + |f_{,1}|^2 + |f_{,2}|^2}$$

examples of norms

f is called L_2 (integrable) if $L_2(f) < \infty$

H_1 if $H_1(f) < \infty$

One can define a norm & show $B(T,T)$ for $\varepsilon = 1$ is coercive

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Some examples of use of coercivity

source term

$$B(\hat{T}, T) = L_i(\hat{T}, Q) + L_u(\hat{T}, \bar{T}) + L_f(\hat{T}, \bar{q}_n)$$

consider T_1 & T_2 satisfy this for Q_1, Q_2 ($\bar{T}$, $\bar{q}_n$ are the same)

we want to see Q_1 & Q_2 are close (forces of the problem)

How close are their corresponding solutions

$$B(\hat{T}, T_1) = L_i(\hat{T}, Q_1) + L_u(\hat{T}, \bar{T}) + L_f(\hat{T}, \bar{q}_n)$$

choose the same $\hat{T}$

$$B(\hat{T}, T_2) = L_i(\hat{T}, Q_2) + L_u(\hat{T}, \bar{T}) + L_f(\hat{T}, \bar{q}_n)$$

subtract

$$B(\hat{T}, T_1 - T_2) = L_i(\hat{T}, Q_1 - Q_2)$$

Choose $\hat{T} = T_1 - T_2$

often assumed continuous on it's arguments

$$\lambda \|T_1 - T_2\|^2 \leq |B(T_1 - T_2, T_1 - T_2)| = |L_i(T_1 - T_2, Q_1 - Q_2)|$$

a norm

coercive

$$\leq C \, \|T_1 - T_2\| \, \|Q_1 - Q_2\|$$

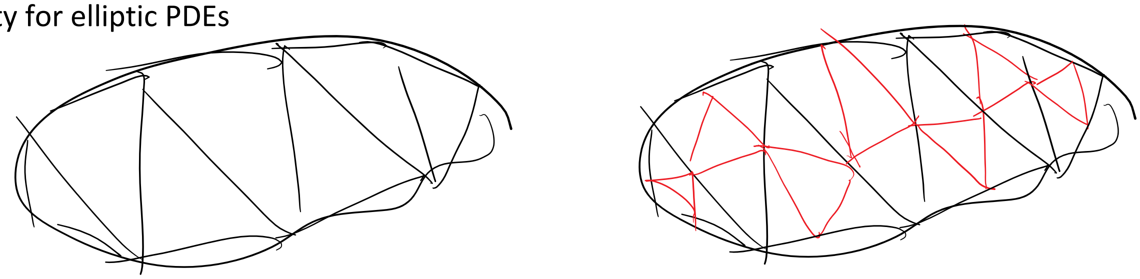
$\exists C$

$$\boxed{\|T_1 - T_2\| \leq \frac{C}{\lambda} \|Q_1 - Q_2\|}$$

as $Q_2 \to Q_1$ so does their corresponding solutions

wellposedness

Stability for elliptic PDEs



solve $Ka=f$

as long as $\det K \neq 0$

we get an a

$h_1 \to \|T^{h_1}\|$

$h_2 \to \|T^{h_2}\|$

often for unstable scheme: $|T| \nearrow$ as $h \searrow$

(the solution becomes more oscillatory as finer elements allow for this

How do we write stability for elliptic PDEs

$\exists\, C_Q, C_{\bar T}, C_{\bar q_n}$ INDEPENDENT of discretization length

$\ni \forall$ discretization

$\bar T$ on $\partial\Omega_u$ $\quad$ $\bar q_n$ on $\partial\Omega_f$

$$\|T^h\| \le \underbrace{C_Q \|Q\|_{\text{inside}} + C_{\bar T}\|\bar T\|_{\partial\Omega_u} + C_{\bar q_n}\|\bar q_n\|_{\partial\Omega_f}}_{\text{constant}}$$

this is a norm

Constant

How does coercivity help us with this property

Solution is obtained from this

$$B(\hat{T},T) = L_i(\tilde{T},Q) + L_u(\hat{T},\bar{T}) + L_f(\hat{T},\bar{q}_n)$$

$\hat{T} = T$

$$|B(T,T)| = |L_i(T,Q) + L_u(T,\bar{T}) + L_f(T,\bar{q}_n)|$$

triangular inequality

$|a+b+c| \leq |a|+|b|+|c|$

$$|B(T,T)| \leq |L_i(T,Q)| + |L_u(T,\bar{T})| + |L_f(T,\bar{q}_n)|$$

just doing it for Q

$$\underbrace{\lambda \|T\|^2 <}_{\text{coercivity}} |B(T,T)| \leq \|T\|\,\|Q\|$$

continuity of L_i ...

$\exists \lambda$ Independent of discretization

$$\|T\| \leq \frac{1}{\lambda}\|Q\|$$

C_Q

norm of T is bounded by the norm of $\|Q\|$

∴ stability

What is the interpretation of coercivity in discrete setting

$$B(\hat{T},T) = L_i(\hat{T},Q)$$

$T = \Phi a \qquad \hat{T} = \Phi \hat{a} \qquad B(\hat{T},T) = \hat{a} K a$

stiffness matrix

$$B(T,T) = a \cdot K a$$

for coercivity we want to find a λ independent of discretization ∋

for coercivity we want to find a λ independent of discretization $\exists$

$$a K a > \lambda |a|^2 > 0$$

$\forall a \quad aKa \geq 0$ positive matrix

$+ \; aKa = 0 \iff a = 0$ pos-def. matrix

necessary condition is that the matrix is positive definite

How do we relate this to eigen values

For discussion assume K is diagonalizable

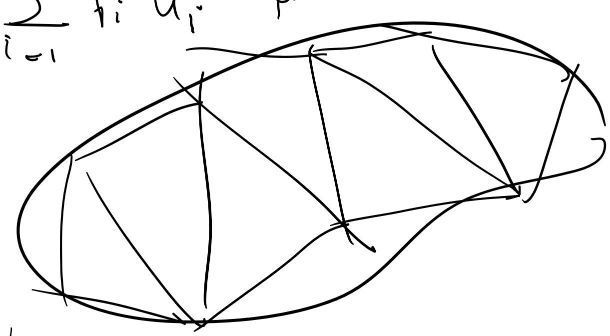
$K u_i = \lambda_i u_i$ for u_i's $\quad u_{(i)} \cdot K u_i = \lambda_i |u_i|^2 \geq 0$

no summation on i

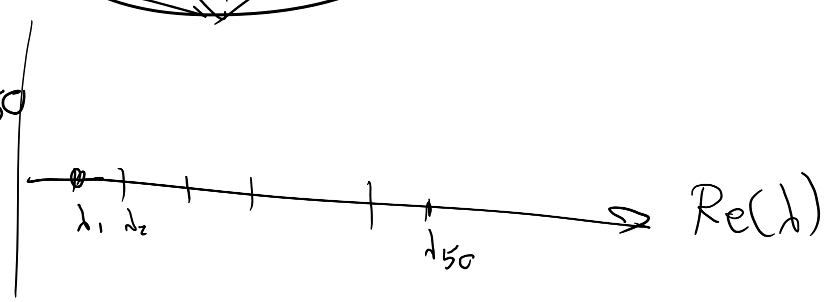
$\lambda_i > 0$

this holds $\quad aKa \geq \lambda |a|^2 \quad$ for $\quad \lambda = \min\{\lambda_1, \ldots, \lambda_n\}$ (*)

$$a = \sum_{i=1}^{n} \beta_i u_i$$ prove that again $aKa \geq \lambda |a|^2$

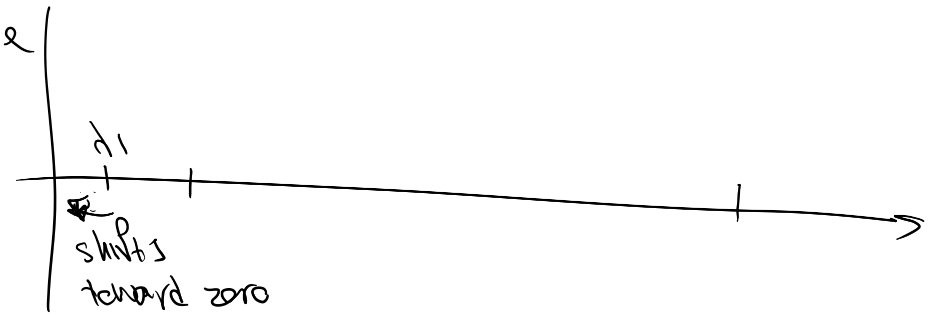


$K_{50 \times 50}$



Refine the mesh

If unstable



200 eigen values

Stable ones

eigen values stay $> \bar{\lambda}$