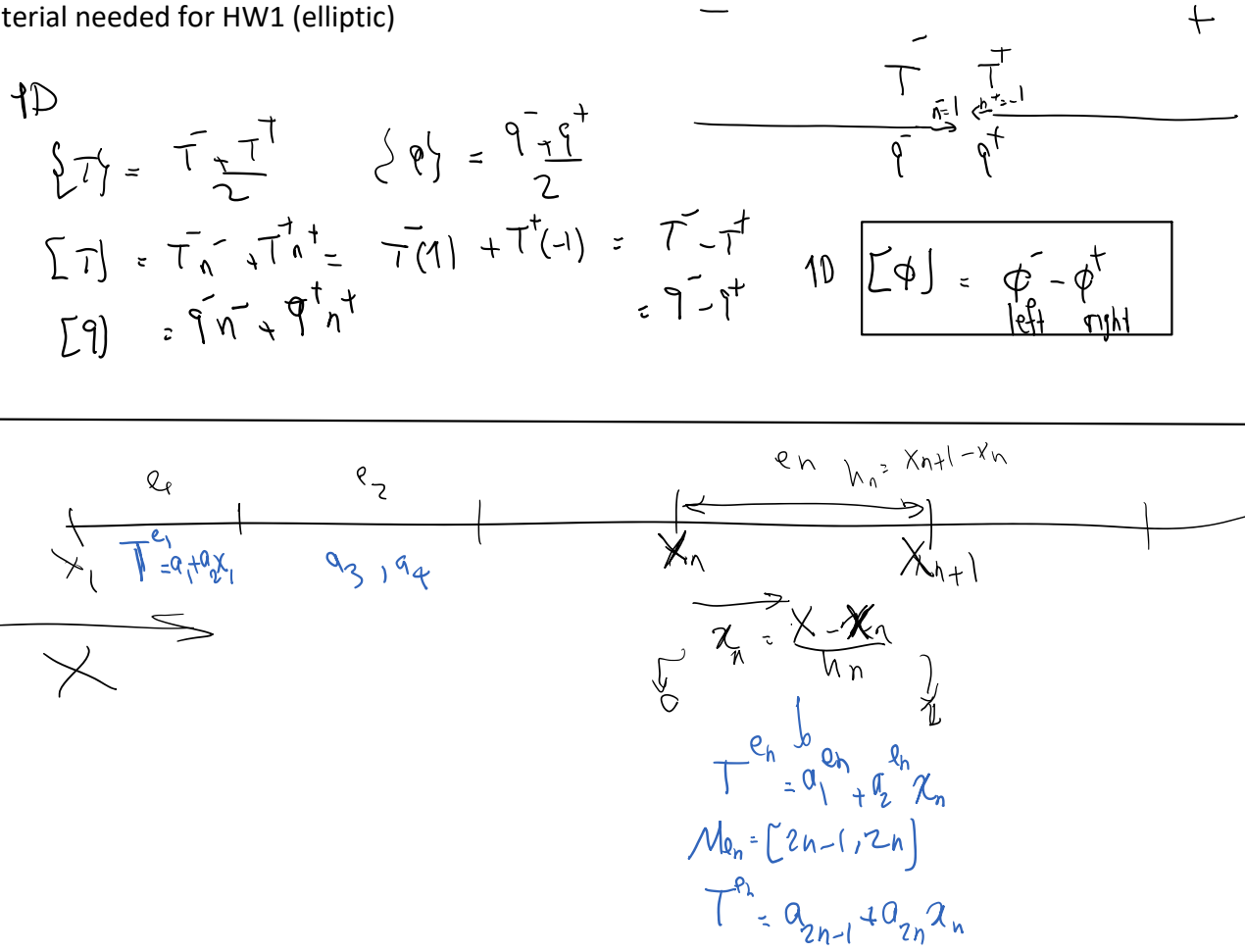
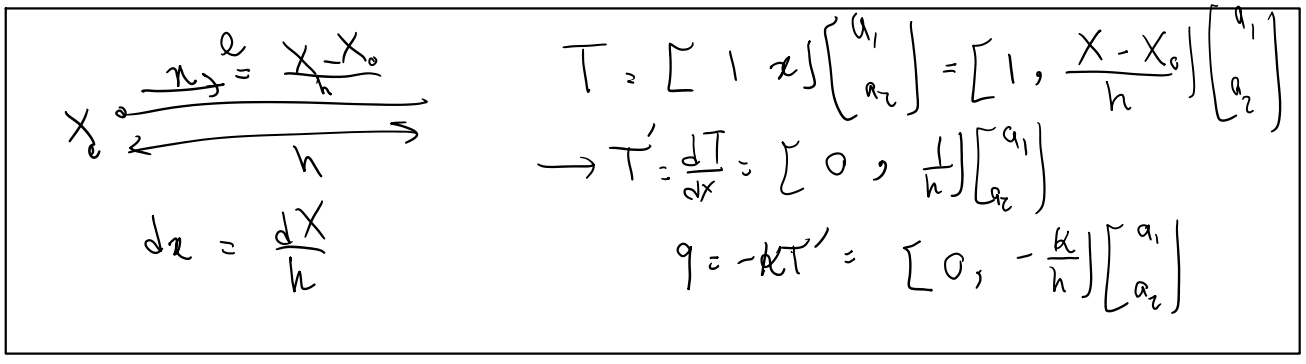
2022/10/13

Thursday, October 13, 2022 9:41 AM

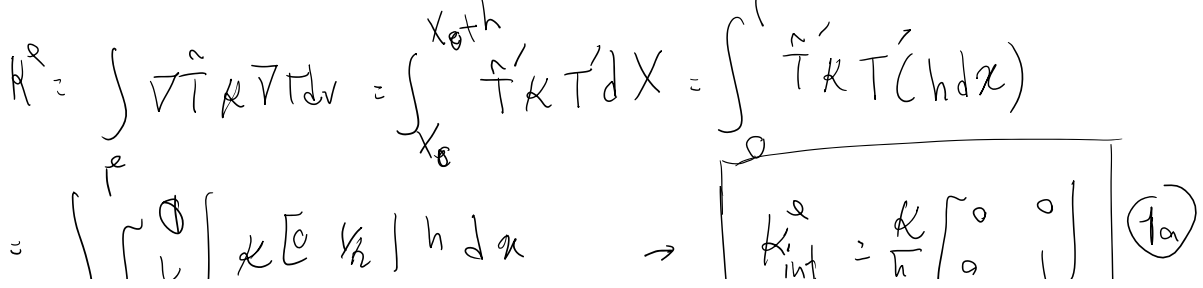
Material needed for HW1 (elliptic)



We want to form the global stiffness matrix and all the forces.



A. Interior of element contributions



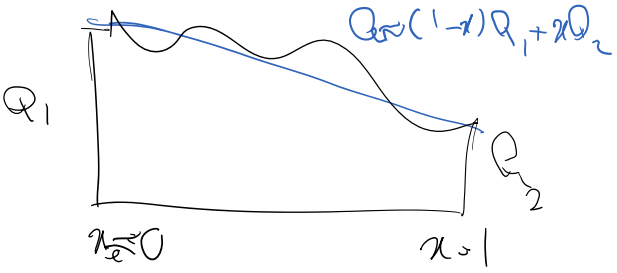
$$= \int_0^1 \begin{bmatrix} 0 \\ \frac{1}{h} \end{bmatrix} k \begin{bmatrix} 0 & \frac{1}{h} \end{bmatrix} h \, dx \rightarrow \boxed{K_{int}^e = \frac{k}{h}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \quad (1a)$$

If we want to solve the parabolic PDE we need to add capacity matrix

$$M = \int \tilde{T} C \dot{T} \, dx = \int_0^1 \begin{bmatrix} 1 \\ x \end{bmatrix} c \begin{bmatrix} 1 & x \end{bmatrix} (h \, dx) \quad \boxed{M^e = ch\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}} \quad (1b)$$

Source term

$$f_Q = \int_{x_0}^{x_0+h} \begin{bmatrix} \pi \\ t_2 \end{bmatrix} Q(x) \, dx =$$



$$= \int_0^1 \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1-x & x \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} (h \, dx) \rightarrow \boxed{f_Q = h\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}} \quad (1c)$$

B. Essential BC:

$$B_u(\tilde{f}, T) = \int_{\partial\Omega_u} \left(\hat{T} q - \varepsilon \nabla \hat{T} q\right) \cdot n \, ds$$

$$L_u(\tilde{f}) = -\varepsilon \int_{\partial\Omega_u} \hat{q}_n \tilde{T} \, ds$$

$\partial\Omega_u \rightarrow$ essential boundary of the element

$$T = \begin{bmatrix} 1 & x \end{bmatrix}$$
$$T' = \begin{bmatrix} 0 & \frac{1}{h} \end{bmatrix}$$
$$q = \begin{bmatrix} 0 & -\frac{k}{h} \end{bmatrix}$$

$$K_u = \left(\hat{T} q - \varepsilon \nabla \hat{T} q\right) \cdot n \Big|_{@\partial\Omega_u} = \left(\begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 0 & -\frac{k}{h} \end{bmatrix} - \varepsilon \begin{bmatrix} 0 \\ \frac{k}{h} \end{bmatrix} \begin{bmatrix} 1 & x \end{bmatrix}\right) \cdot n$$

$$F_u = -\varepsilon \hat{q}_n \tilde{T} \Big|_{@\partial\Omega_u} = -\varepsilon \begin{bmatrix} 0 \\ -\frac{k}{h} \end{bmatrix} \vec{n} \tilde{T}$$

$$K_u^L = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{k}{h} \end{bmatrix} - \varepsilon \begin{bmatrix} 0 & -\frac{k}{h} \end{bmatrix}\right)(-1) \qquad K_u^R = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{k}{h} \end{bmatrix} - \varepsilon \begin{bmatrix} 0 \\ -\frac{k}{h} \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

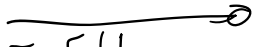
$K_u^L = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{\alpha}{h} \end{bmatrix} - \varepsilon \begin{bmatrix} 0 & -\frac{\alpha}{h} \end{bmatrix} \right)(-1)$ $\quad K_u^R = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{\alpha}{h} \end{bmatrix} - \varepsilon \begin{bmatrix} 0 \\ -\frac{\alpha}{h} \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$

$f_u^L = -\varepsilon(-1)\bar{T} \begin{bmatrix} 0 \\ -\frac{\alpha}{h} \end{bmatrix}$ $\quad f_u^R = -\varepsilon \begin{bmatrix} 0 \\ -\frac{\alpha}{h} \end{bmatrix} (1)\bar{T}$

C. Natural BC $\quad B_\beta(\hat{T}, T) = 0$

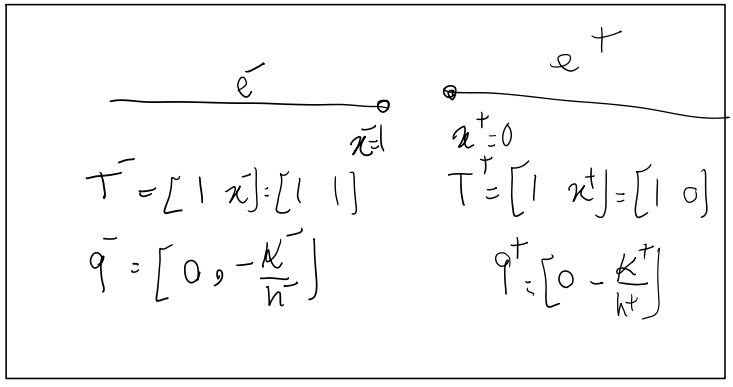
$h_\beta(\hat{T}) = \int_{\partial\Omega_\beta} \hat{T} \bar{q}_n \, ds \quad \rightarrow \quad F_\beta = \begin{bmatrix} 1 \\ x \end{bmatrix} q_n$ @ natural Boundary

$F_\beta = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{q}_n$



$F_\beta = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bar{q}_n$

D. K and F terms from interior interfaces



$\int_{\partial e^+} [\hat{T}](\{q\} + \bar{\alpha}[T]) - \varepsilon\{\bar{q}\}[T] \, ds$

$[T]$ → row vector $[\hat{T}] = [T]^t$ column wise

$\{q\} = \{\hat{q}\} = \{q\}^t$

$[T] = T^- - T^+ = e^- \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^+ \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

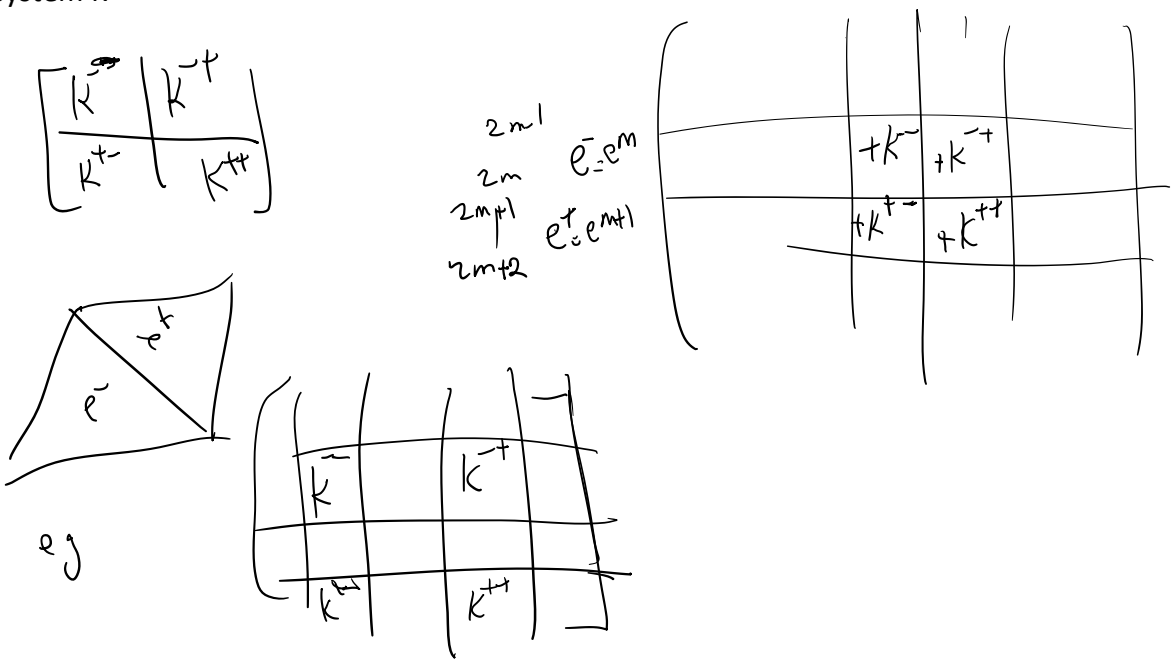
$\{q\} = \frac{\hat{q}^- + \hat{q}^+}{2} = \frac{1}{2}\left(\begin{bmatrix} 0 \\ -\frac{K^-}{h^-} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{K^+}{h^+} \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -\frac{K^-}{2h^-} \\ 0 \\ -\frac{K^+}{2h^+} \end{bmatrix}$

$\int_{\partial e^+} [\hat{T}]\{q\} + \bar{\alpha}[\hat{T}][T] - \varepsilon\{\bar{q}\}[T]$

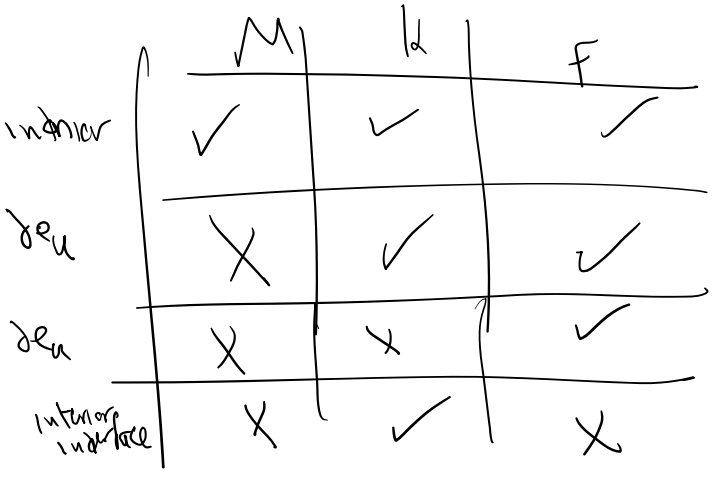
ⓐ ⓑ ⓒ

ⓐ $[T]\{q\} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{K^-}{2h^-} & 0 & -\frac{K^+}{2h^+} \end{bmatrix}$, ⓑ $\bar{\alpha}[\hat{T}][T] = \bar{\alpha} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}$

By adding terms a), b), c) we get a 4x4 interface stiffness matrix that needs to subsequently assembled to the global system K



After the assembly, we get K and F (and M for parabolic PDE)



Parabolic Case (HW2), only the mass (capacity) matrix is added
We'll use a forward Euler scheme

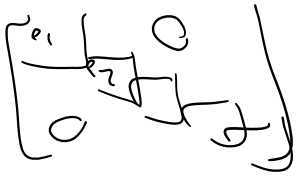
$$M\ddot{a}_n + Ka_n = F_n$$

$$M\left(\frac{a_{n+1} - a_n}{\Delta t}\right) + Ka_n = F_n$$

$$M a_{n+1} = M a_n - \Delta t K a_n - \Delta t F$$

M: block diagonal

we can directly add element contribution



$$M_e a^e_{n+1} = F^e$$

In practice, we don't form a global C, and only solve the equations at the element level.
We basically solve 1 element at a time. -> Ideal for parallel computing
As noted, we never form any stiffness matrix for explicit schemes.

Time step for parbolic PDEs

$$C\dot{T} - (kT')' = Q$$

assume k const

$$\dot{T} - \gamma T_{xx} = Q/c$$

Diffusion coef $\frac{L^2}{T}$

$$\Delta t \propto C_p \frac{h_e^2}{2D} \qquad h_e = \frac{h}{p+1}$$

polynomial dependent coefficient & time marching

Some points about elliptic and parabolic PDEs

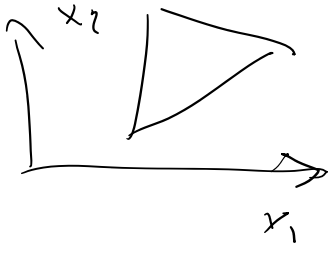
1. Single versus two-field formulations

PDE $\quad C\dot{T} + \nabla \cdot q = Q$

2-field formulation

$$T = a_1 + a_2 x_1 + a_3 x_2$$

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} a_4 + a_5 x_1 + a_6 x_2 \\ a_7 + a_8 x_1 + a_9 x_2 \end{bmatrix}$$



Interpolating vector q significantly increases the number of global unkowns in 2D and 3D

We need to add a compatibility condition between q and T

$$k\nabla T = -q \quad \rightarrow \quad \nabla T = -k^{-1} q$$

WRS

$$\int_e \hat{T}(c\dot{T} + \nabla\cdot q - Q)\,dV + \int_{\partial e} \hat{T}(q_n^0 - q\cdot n)\,ds$$
$$+\varepsilon\left(\int_e \hat{q}\,(\nabla T + k^{-1} q)\,dV + \int_{\partial e} \hat{q}\cdot n\,(T^0 - T)\,ds \right)$$

($\nabla T + k^{-1} q$: compatibility (constitutive eqn))

We can solve for T and q as independent fields by assembling the global K and F and solving for the unknowns of T and q.

There are 2 approaches to only solve for T at the global level.

Approach a:

Don't interpolate T $\quad T = a_1 + a_2 x_1 + a_3 x_2 \quad$ $q = -k\nabla T$ is directly obtained from T. as we did in prev. session

1F:

$$\int_e \hat{T}(c\dot{T} + \nabla\cdot q - Q)\,dV + \int_{\partial e} \hat{T}(q_n^0 - q\cdot n)\,ds$$
$$+\varepsilon\left(\int_e \hat{q}\,(\nabla T + k^{-1} q)\,dV + \int_{\partial e} \hat{q}\cdot n\,(T^0 - T)\,ds \right)$$

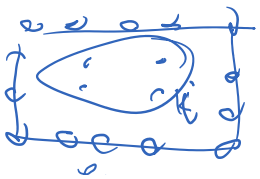
($\nabla T + k^{-1} q$: compatibility (constitutive eqn))

So we're back to 1F formulation we had before

$$\int_e \hat{T}(c\dot{T} + \nabla\cdot q - Q)\,dV + \int_{\partial e} \hat{T}(q_n^0 - q\cdot n)\,ds$$
$$+\varepsilon \int_{\partial e} \hat{q}\cdot n\,(T^0 - T)\,ds = 0$$

Approach b: Remove q dofs from the global system

In this case, we have a genuine 2F formulation but at the global level, we only solve for T, go back to the elements and solve q's



$$k_{ee}\,a_e + k_{ei}\,a_i = f_e$$
$$k_{ie}\,a_e + k_{ii}\,a_i = f_i$$

$K_{ie} a_e + K_{ii} a_i = f_i$

$\rightarrow \tilde{K}_{ee} a_e = \tilde{f}_e$ $\quad \tilde{K} = K_{ee} - K_{ei} K_{ii}^{-1} K_{ie}$

$\tilde{f}_e = \tilde{f}_e - K_{ei} K_{ii}^{-1} f_i$

We need to apply condensation to q dofs and remove them from element dofs when assembling to the global K

$q^* = [q] + \beta [q] + \alpha [T]$

$T^* = [T] + \partial [T] + \delta [q]$

not having this term allows us to condense q out

Why do we want to have a 2F formulation anyway

reason i

$T \rightarrow p^{th}$ polynomial $\quad \| T - T^{exact} \| \leq C h^{p+1}$

L^2 norm — optimal convergence rate

$q = -k \nabla T$

1F $\quad \| q - q^{exact} \| \leq C h^{p}$

1 der for q

2F q is interpolated

directly say again with p^{th} order $\quad \| q - q^{exact} \| \leq C h^{p+1}$

reason ii) Stability & limit on $\tilde{\alpha} = \alpha k$, $\alpha = \frac{n_0}{h}$

Arnold 2002

TABLE 3.1
Some DG methods and their numerical fluxes.

Method	$\widehat{u}_K$	$\widehat{\sigma}_K$
Bassi Rebay [9]	$\{u_h\}$	$\{\sigma_h\}$
Brezzi et al. [18]	$\{u_h\}$	$\{\sigma_h\} - \alpha_r([\![u_h]\!])$
LDG [35]	$\{u_h\} - \beta \cdot [\![u_h]\!]$	$\{\sigma_h\} + \beta[\![\sigma_h]\!] - \alpha_j([\![u_h]\!])$
IP [43]	$\{u_h\}$	$\{\nabla_h u_h\} - \alpha_j([\![u_h]\!])$
Bassi et al. [10]	$\{u_h\}$	$\{\nabla_h u_h\} - \alpha_r([\![u_h]\!])$
Baumann Oden [12]	$\{u_h\} + n_K \cdot [\![u_h]\!]$	$\{\nabla_h u_h\}$
NIPG [53]	$\{u_h\} + n_K \cdot [\![u_h]\!]$	$\{\nabla_h u_h\} - \alpha_j([\![u_h]\!])$
Babuška Zlámal [6]	$(u_h\|_K)\|_{\partial K}$	$-\alpha_j([\![u_h]\!])$
Brezzi et al. [19]	$(u_h\|_K)\|_{\partial K}$	$-\alpha_r([\![u_h]\!])$

2F T, σ interpolated

1F $T, \sigma = \nabla T$

σ is ∇u in this paper

Babuška Zlámal [6]	$(u_h\|_K)\|_{\partial K}$	$-\alpha_j([\![u_h]\!])$
Brezzi et al. [19]	$(u_h\|_K)\|_{\partial K}$	$-\alpha_r([\![u_h]\!])$

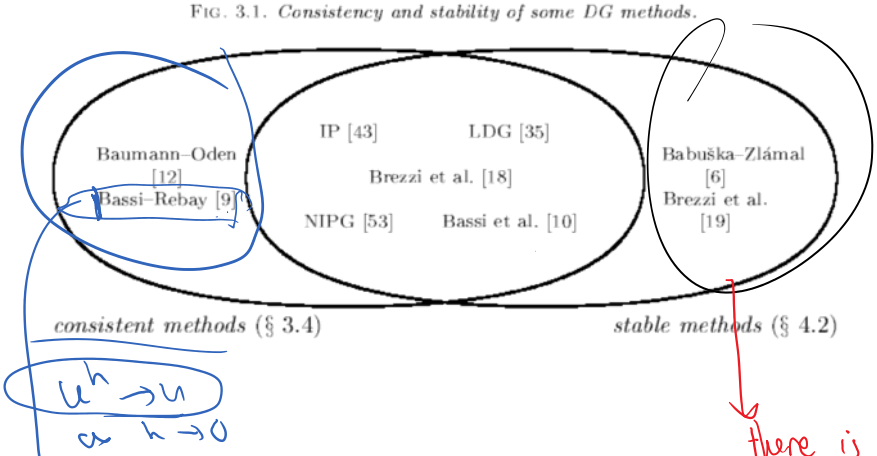
TABLE 6.1
Properties of the DG methods

Method	cons.	a.c.	stab.	type	cond.	H^1	L^2
Brezzi et al. [18]	✓	✓	✓	α^r	$\eta_0 > 0$	h^p	h^{p+1}
LDG [35]	✓	✓	✓	α^j	$\eta_0 > 0$	h^p	h^{p+1}
IP [43]	✓	✓	✓	α^j	$\eta_0 > \eta^*$	h^p	h^{p+1}
Bassi et al. [10]	✓	✓	✓	α^r	$\eta_0 > 3$	h^p	h^{p+1}
NIPG [53]	✓	×	✓	α^j	$\eta_0 > 0$	h^p	h^p
Babuška Zlámal [6]	×	×	✓	α^j	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Brezzi et al. [19]	×	×	✓	α^r	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Baumann Oden ($p = 1$)	✓	×	×	-	-	×	×
Baumann Oden ($p \geq 2$)	✓	×	×	-	-	h^p	h^p
Bassi Rebay [9]	✓	✓	×	-	-	$[h^p]$	$[h^{p+1}]$

Hybridizable DG methods (HDG) is a relatively new DG method, whose dofs (unknows) are the target values on element interfaces (i.e. they are already at the skeleton of the mesh). Accordingly getting rid of "interior dofs" is much easier (like CFEM) and there is no need to have l and r operators of the LDG method to get rid of q dofs.

Discussing this the next time

FIG. 3.1. *Consistency and stability of some DG methods.*



Solution does not blow up (ie is bounded) as $h \to 0$
But it does not converge to the right solution

$u^h \to u$ as $h \to 0$

there is a trick to make them consistent ($u^h \to u^{exact}$ as $h \to 0$)

It's weakly stable: for some meshes (structured) the scheme is not stable