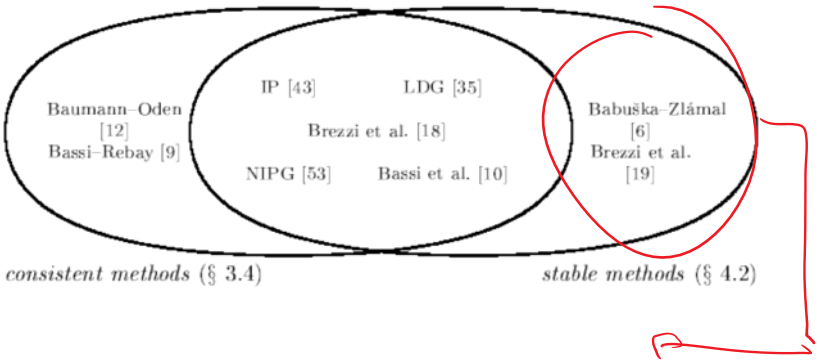
2022/10/18

Tuesday, October 18, 2022 9:44 AM

From last time

FIG. 3.1. *Consistency and stability of some DG methods.*



How to make these schemes stable?

For these unstable methods, we can choose the penalty (alpha) large enough to make them stable.

But the large penalty messes up with the condition number of the system matrix

TABLE 6.1
Properties of the DG methods

Method	cons.	a.c.	stab.	type	cond.	H^1	L^2
Brezzi et al. [18]	✓	✓	✓	α^r	$\eta_0 > 0$	h^p	h^{p+1}
LDG [35]	✓	✓	✓	α^j	$\eta_0 > 0$	h^p	h^{p+1}
IP [43]	✓	✓	✓	α^j	$\eta_0 > \eta^*$	h^p	h^{p+1}
Bassi et al. [10]	✓	✓	✓	α^r	$\eta_0 > 3$	h^p	h^{p+1}
NIPG [53]	✓	×	✓	α^j	$\eta_0 > 0$	h^p	h^p
Babuška–Zlámal [6]	×	×	✓	α^j	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Brezzi et al. [19]	×	×	✓	α^r	$\eta_0 \approx h^{-2p}$	h^p	h^{p+1}
Baumann–Oden ($p = 1$)	✓	×	×	-	-	×	×
Baumann–Oden ($p \geq 2$)	✓	×	×	-	-	h^p	h^p
Bassi–Rebay [9]	✓	✓	×	-	-	$[h^p]$	$[h^{p+1}]$

penalty grows very fast versus h

PERFORMANCE OF DISCONTINUOUS GALERKIN METHODS FOR ELLIPTIC PDE'S

PAUL CASTILLO *

Abstract. In this paper, we compare the performance of the main discontinuous Galerkin (DG) methods for elliptic partial differential equations on a model problem. Theoretical estimates of the condition number of the stiffness matrix are given for DG methods whose bilinear form is symmetric, which are shown to be sharp numerically. Then, the efficiency of the methods in the computation of both the potential and its gradient is tested on unstructured triangular meshes.

TABLE 5.2
Asymptotic behavior of the spectral condition number $\kappa(h)$ as a function of the mesh size, when using uniform approximations of degree p.

method	penalization	$\kappa(h)$
Babuška-Zlámal	$O\left(h^{-(2p+1)}\right)$	$O\left(h^{-(2p+2)}\right)$

fast growing penalty

method	penalization	$\kappa(h)$
Babuška-Zlámal	$O\left(h^{-(2p+1)}\right)$	$O\left(h^{-(2p+2)}\right)$
IP	$O\left(h^{-1}\right)$	$O\left(h^{-2}\right)$
LDG	$O\left(h^{-1}\right)$	$O\left(h^{-2}\right)$
Baumann-Oden	no penalization	$O\left(h^{-2}\right)$
NIPG1	$O\left(h^{-1}\right)$	$O\left(h^{-2}\right)$
NIPG3	$O\left(h^{-3}\right)$	$O\left(h^{-4}\right)$

fast growing penalty makes the condition number grow faster versus h

decent similar to CFEM

From all the previous discussions, it appears that LDG method is the best option (optimal convergence rate -> see below) and eta_0 just needs to be positive.

The downside is that it's a two field formulation, but as discussed last time by using lift operators, r and l and using the fact that T* is only a function of T, we can condense q out from the global system

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LDG fluxes: central versus alternating flux options

$$q^* = \{q\} + \vec{\beta}[q] + \tilde{\alpha}[T]$$
$$T^* = \{T\} + \vec{\gamma}[T] \qquad \vec{\gamma} = -\vec{\beta}$$

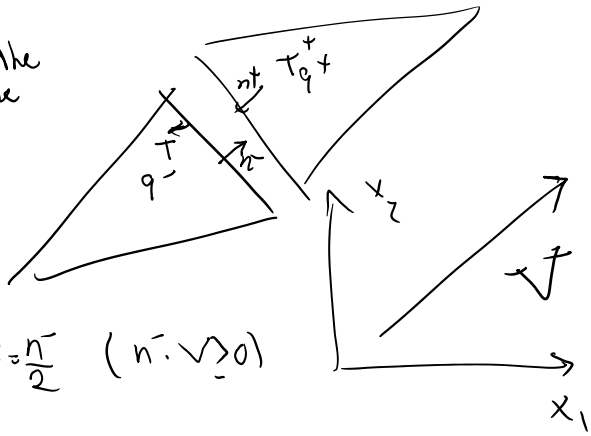
Recall $\vec{\beta} = 0$ Resulted in central flux

$$q^* = \{q\} + \tilde{\alpha}[T]$$
$$T^* = \{T\}$$

Alternating flux option:

$$\vec{\beta} = \begin{cases} \frac{n^-}{2} & \text{if } n^- \cdot V \geq 0 \\ \frac{n^+}{2} & \text{else} \end{cases} \qquad \vec{\gamma} = \vec{\beta}$$

like in the figure



$$q^* = \{q\} + \vec{\beta}[q] + \tilde{\alpha}[T]$$
$$= \left(\frac{q^- + q^+}{2}\right) + \frac{n^-}{2}\left(n^- q^- + n^+ q^+\right) + \tilde{\alpha}[T] \quad \text{if } \beta = \frac{n^-}{2} \quad (n^- \cdot V \geq 0)$$
$$= \frac{q^- + q^+}{2} + \left(\frac{q^-}{2} - \frac{q^+}{2}\right) + \tilde{\alpha}[T]$$

$n^- n^- = 1 \quad n^- n^+ = -1$

$$q^* = q^- + \tilde{\alpha}[T]$$

$$T^* = \{T\} + \vec{\beta}[T] = \left(\frac{T^- + T^+}{2}\right) - \frac{n^-}{2}\left(T^- n^- + T^+ n^+\right) = \left(T^- + \frac{T^- - T^+}{2}\right) = \frac{T^-}{2} + \frac{T^+}{2}$$

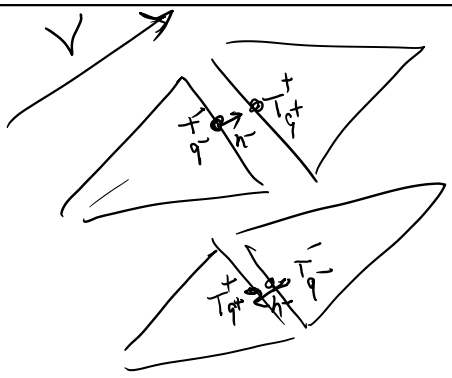
$$\hat{T}^* = \{T\} + \beta[T] = \left(\frac{T^- + T^+}{2}\right) - \frac{n^-}{2}\left(T^- n^- + T^+ n^+\right) = \left(\frac{T^- + T^+}{2}\right) - \frac{T^-}{2} + \frac{T^-}{2}$$

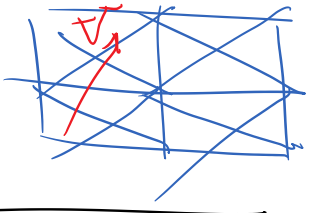
$$\hat{T}^* = T^+ \quad -\beta$$

else case $\left(\vec{\beta} = \frac{n^+}{2} \quad (\bar{n}\cdot v < 0)\right)$ $\quad q^* = q^+ - \alpha[T]$, $\hat{T}^* = T^-$

$$\begin{cases} q^* = q^- + \alpha[T] \\ \hat{T}^* = T^+ \end{cases}$$

$$\begin{cases} q^* = q^+ + \alpha[T] \\ \hat{T}^* = T^- \end{cases}$$

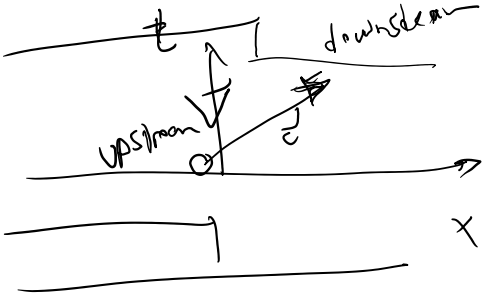




0, 45, 90 don't choose these as $\vec{v}$

The idea is from hyperbolic PDEs

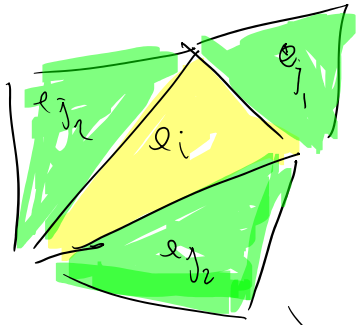
$u_t + c u_x = 0$



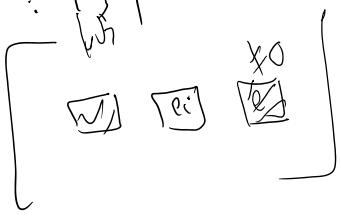
Which one is better?

LDG condensing q out

central flux

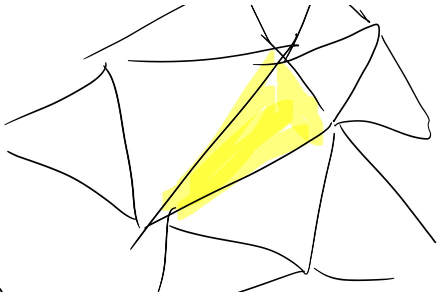


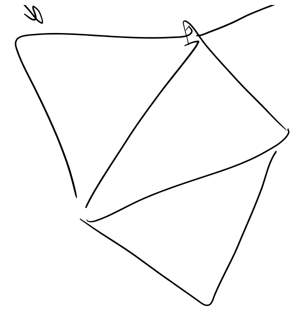
$\mathcal{F}: T, q$



similar

alternating flux does not expand





does not expand the stencil

stencil expands with central difference

DIFFERENT FORMULATIONS OF THE DISCONTINUOUS GALERKIN METHOD FOR THE VISCOUS TERMS*

CHI-WANG SHU†

Abstract. Discontinuous Galerkin method is a finite element method using completely discontinuous piecewise polynomial space for the numerical solution and the test functions. Until recently it was mainly used for solving convection problems involving only first spatial derivatives. Recently the method has been extended successfully to solve convection diffusion problems involving second derivative viscous terms. In this paper we will use simple examples to illustrate the basic ideas and "pitfalls" for using the discontinuous Galerkin method on the viscous terms.

4. The local discontinuous Galerkin method for the second order diffusion problem. If we rewrite the heat equation (3.1) as a first order system

$$u_t - q_x = 0, \qquad q - u_x = 0, \tag{4.1}$$

$$\hat{u}_{j+\frac{1}{2}} = \frac{1}{2}\left(u^-_{j+\frac{1}{2}} + u^+_{j+\frac{1}{2}}\right), \qquad \hat{q}_{j+\frac{1}{2}} = \frac{1}{2}\left(q^-_{j+\frac{1}{2}} + q^+_{j+\frac{1}{2}}\right). \tag{4.3}$$

2. The order of accuracy is one order lower for odd k. That is, for odd k the proof of the sub-optimal error estimate of order k is actually sharp.

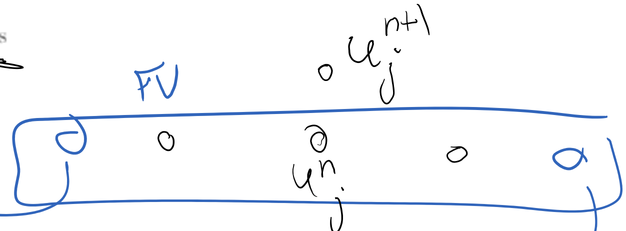
for element order odd convergence rate is by 1 suboptimal

P2

optimal T (U) → order k element error(T) ∝ Ch^{k+1} → don't have 1 for odd k

1. It spreads to five cells when a local basis is chosen for u in cell I_j. After q is eliminated the scheme becomes

$$\frac{d}{dt}u_j + \frac{1}{\Delta x_j^2}\left(Au_{j-2} + Bu_{j-1} + Cu_j + Du_{j+1} + Eu_{j+2}\right) = 0$$



expanded stencil after condensing q

P4

as for central flux we have 1 additional layer connectivity

Alternating flux:

P1 -> eliminated -> we have optimal convergence for any order of element.
P2 -> just one layer connectivity even with q condensation

Both problems can be cured by a clever choice of fluxes, proposed in Cockburn and Shu [8]:

(4.4) $\hat{u}_{j+\frac{1}{2}} = u^-_{j+\frac{1}{2}}, \quad \hat{q}_{j+\frac{1}{2}} = q^+_{j+\frac{1}{2}}.$

(3.3), even though we now have nominally an additional auxiliary variable q! We can also prove that now the order of accuracy becomes $k+1$ for all k.

$$\frac{d}{dt}u_j + \frac{1}{\Delta x_j^2}(Au_{j-2} + Bu_{j-1} + Cu_j + Du_{j+1} + Eu_{j+2}) = 0$$

Remaining for parabolic case:
- erroneous parabolic flux DG_course\Papers\Fluxes \Elliptic+Parabolic\ErroneousParabolicFluxes
- A more physical way to get parabolic star values: Lorcher_2008_An explicit discontinuous Galerkin scheme with local time-stepping for general unsteady diffusion equations

Hyperbolic PDEs

Scalar wave equation , u is scalar

$$\rho \ddot{u} + d\dot{u} - \nabla\cdot(k\nabla u) = S \quad \text{(1)}$$

S: source term

Elastodynamics

$u \rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $\qquad k\nabla u \rightarrow \sigma(\varepsilon) \qquad \varepsilon = \frac{\nabla u + \nabla u^T}{2}$

Write (1) as a system of conservation laws:

$$\begin{cases} v = \dot{u} & \text{velocity} \\ q = \nabla u & \text{"strain-like" quantity} \end{cases}$$

$$\dot{U} + F_{x,x}(U) + F_{y,y}(U) + F_{z,z}(U) = S$$

$$F = [F_x | F_y | F_z]$$

$$\boxed{\dot{U} + \nabla \cdot F = S}$$ Conservation law form

For semi-linear PDE, F is a linear function of U

$$\underbrace{\rho \ddot{u}}_{\text{time}} + d\dot{u} - \underbrace{\nabla \cdot (k \nabla u)}_{\text{space}} = s$$

$$\underbrace{\rho \dot{v}}_{\text{time}} + d\dot{v} - \underbrace{\nabla \cdot (kq)}_{\text{space}} = s$$

$$\dot{u} - v = 0$$

$$\dot{q} - \nabla v = 0 \quad \longleftarrow \quad (q = \nabla u)$$

$(\dot{\ })$ $\nabla \cdot / \nabla$

$$\boxed{\begin{array}{lll} \dot{U} & & = V \\ \rho \dot{V} & -\nabla \cdot kq & = -dV + s \\ \dot{q} & -\nabla V & = 0 \end{array}}$$

$\nabla \cdot IV$

system of equations

$$U = \begin{bmatrix} u \\ \rho v \\ q \end{bmatrix} \qquad F = \begin{bmatrix} 0 \\ -kq \\ vI \end{bmatrix} \qquad S = \begin{bmatrix} v \\ -dv + s \\ 0 \end{bmatrix}$$

$$\boxed{\dot{U} + \nabla \cdot F = S}$$ side note we can write this

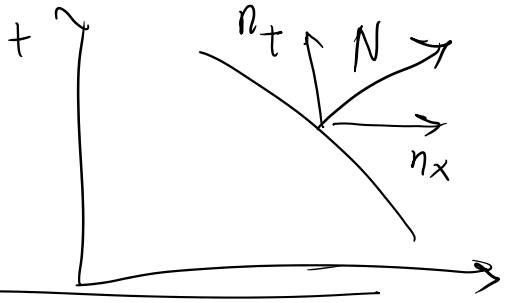
$$\dot{U} + \nabla \cdot F = S$$

side note we can write this as a space-time flux

$$M = \begin{bmatrix} F(U) \\ U \end{bmatrix}$$

$$\nabla_{st} \cdot M = S$$

$$\rightarrow \nabla_S \cdot F(U) + \frac{\partial}{\partial t} U = S$$

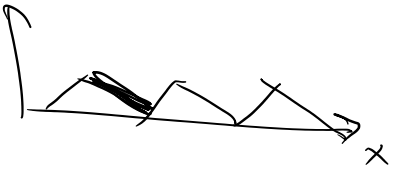


$$[M] \cdot N = 0$$

$$\begin{bmatrix} F(U) \\ U \end{bmatrix} \cdot \begin{bmatrix} \vec{n}_x \\ n_t \end{bmatrix} = 0$$

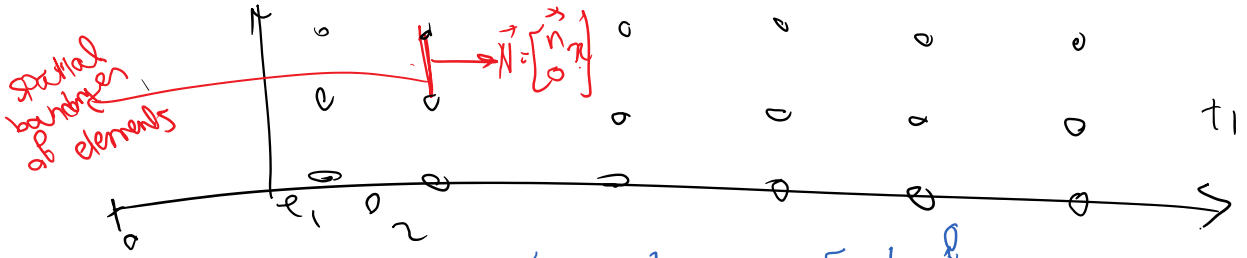
$$\boxed{[F(U)] \cdot n_x + [U] \cdot n_t = 0} \quad (*)$$

general form of jump condition across a manifold t



(*) is useful for space-time DG method

for space DG + time marching



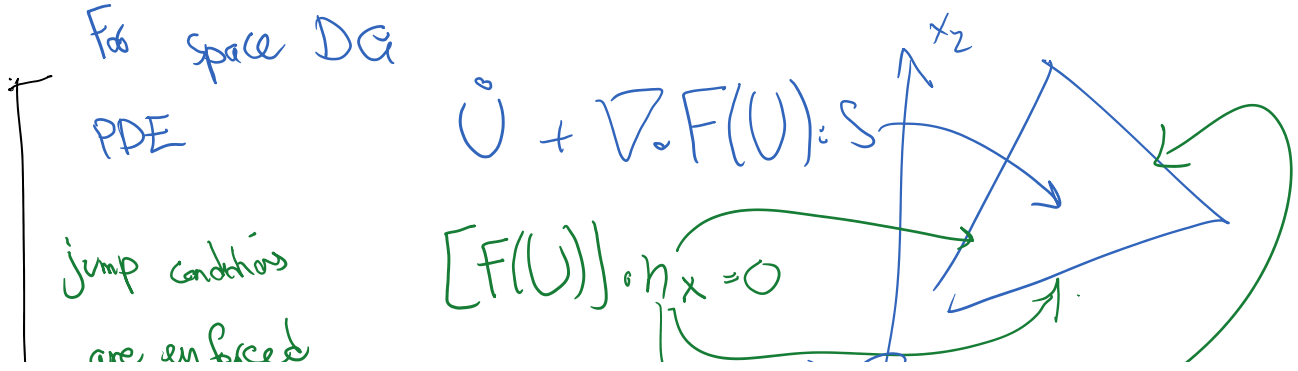
$$[F(U)] \cdot n_x + [U] \cdot \vec{n}_t = 0$$

For space DG PDE

$$\dot{U} + \nabla \cdot F(U) = S$$

jump conditions are enforced

$$[F(U)] \cdot n_x = 0$$



Jump conditions

are enforced

on the boundary of element

$[\![u]\!] \cdot n_x = 0$