$$f^e = \underset{\text{Source}}{f_r} + \underset{\text{Neumann}}{f_q^e} - \underset{\text{Dirichlet}}{f_D}$$

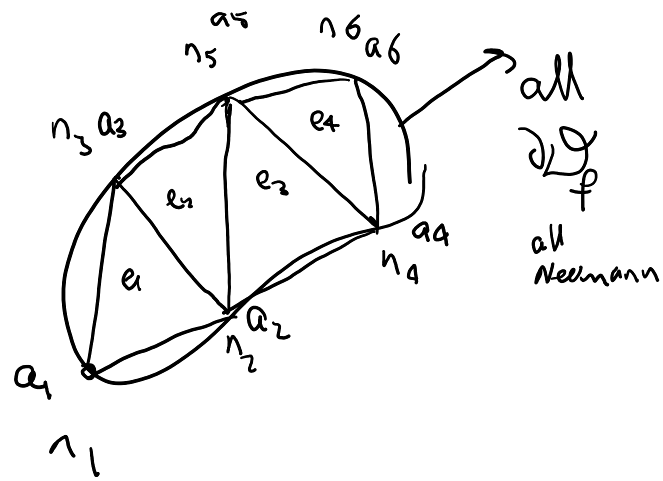
$n_f = 6$

$n_p = 0 \quad (\ldots)$

$$F_r = \iint_{\Omega} \begin{bmatrix} N_1 \\ \vdots \\ N_6 \end{bmatrix} Q \, dV$$

Q → heat conduction problem:

$r : Q$



2D heat conduction

T (temperature) unknown

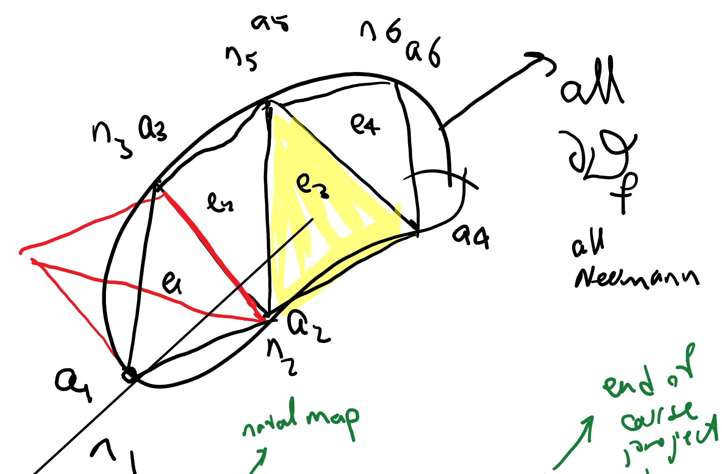
1 dof / node

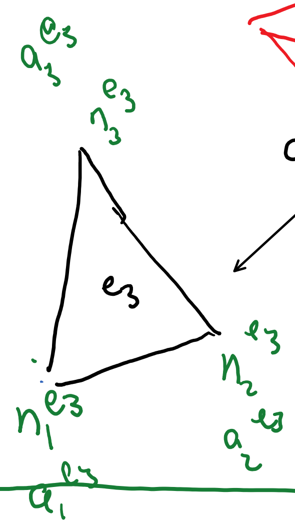
$$= \underbrace{\int_{e_1} N^T Q \, dV}_{F_r^{e_1}} \ldots + \int_{e_4} N^T Q \, dV$$

$$F_r^{e_3} = \int_{e_3} \begin{bmatrix} \cancel{N_1} \to 0 \\ N_2 \\ \cancel{N_3} \\ N_4 \\ \cancel{N_6} \to 0 \end{bmatrix} Q \, dV$$

0

$$= \int_{e_3} \begin{bmatrix} 0 \\ N_2 \\ 0 \\ N_4 \\ N_5 \\ 0 \end{bmatrix} Q \, dV$$





nodal map

$LEM_{e_3} = [2, 4, 5]$ → end of course project text input

$M_{e_3} = [2, 4, 5]$

dof Map

$$= \int_{e_3} \begin{bmatrix} 0 \\ N_1^{e_3} \\ 0 \\ N_2^{e_3} \\ N_3^{e_3} \\ 0 \end{bmatrix} Q \, dV$$

$$f_r^{e_3} = \int \begin{bmatrix} N_1^{e_3} \\ N_2^{e_3} \\ N_3^{e_3} \end{bmatrix} Q \, dV = \begin{array}{c} 2 \\ 4 \\ 5 \end{array} \begin{bmatrix} \int_{e_3} N_1^{e_3} Q \, dV \\ \int N_2^{e_3} Q \, dV \\ \int N_3^{e_3} Q \, dV \end{bmatrix}$$

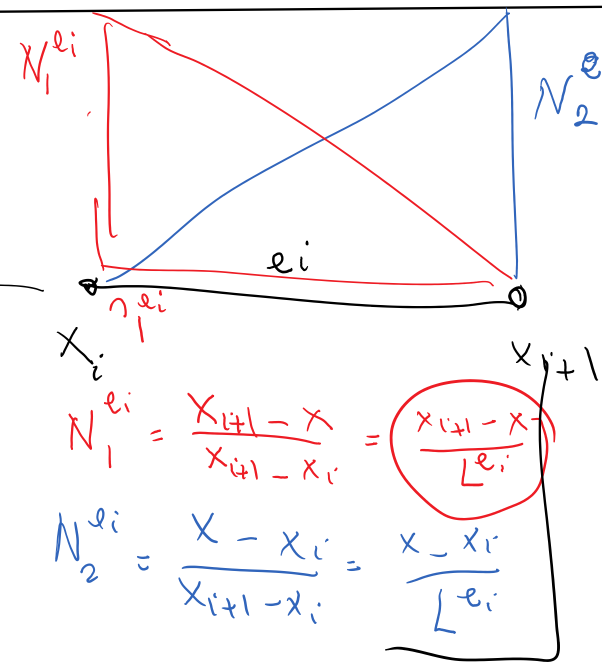
$$\begin{bmatrix} 0 \\ \int_{e_3} N_1^{e_3} Q \, dV \\ 0 \\ \int_{e_3} N_2^{e_3} Q \, dV \\ \int_{e_3} N_3^{e_3} Q \, dV \\ 0 \end{bmatrix}$$

$\int [N_2^{e_3}]$ $\quad \int [\int N_3^{e_3} q\,dv] \rightarrow \begin{bmatrix} \int N_3^{e_3} q\,dv \\ 0 \end{bmatrix}$

the local f_p^e assembled to global F_r matches contributions of element e

---

There were two other formulas I used last time:



1. Element stiffness matrix

$$k^e = \int B^{e^t} D B^e dv$$

$$k^{e_i} = \int_{x_i}^{x_{i+1}} \begin{bmatrix} B_1^{e_i} \\ B_2^{e_i} \end{bmatrix} (EA)^{e_i} [B_1^{e_i} \quad B_2^{e_i}]\, dx$$

if EA is constant, it goes out of the integral

$$N_1^{e_i} = \frac{x_{i+1} - x}{x_{i+1} - x_i} = \frac{x_{i+1} - x}{L^{e_i}} \qquad L^{e_i} = x_{i+1} - x_i$$

$$N_2^{e_i} = \frac{x - x_i}{x_{i+1} - x_i} = \frac{x - x_i}{L^{e_i}}$$

$$B^{e_i} = [B_1^{e_i} \quad B_2^{e_i}] = \underbrace{\frac{d}{dx}}_{Lm} [N_1^{e_i} \quad N_2^{e_i}] = \left[-\frac{1}{L^{e_i}} \quad \frac{1}{L^{e_i}}\right]$$

$$k^e = \int_{x_i}^{x_{i+1}} \begin{bmatrix} -\frac{1}{L^{e_i}} \\ \frac{1}{L^{e_i}} \end{bmatrix} \underbrace{(EA)^{e_i}}_{\text{constant assumed}} \left[-\frac{1}{L^{e_i}} \quad \frac{1}{L^{e_i}}\right] dx$$

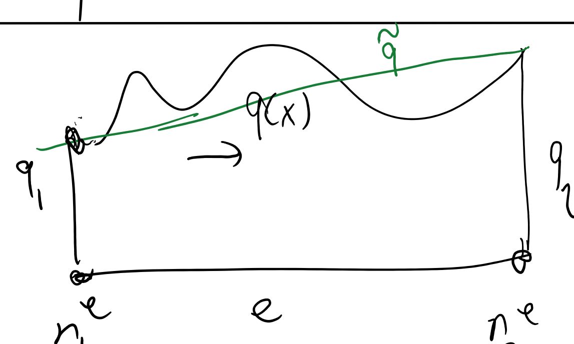
$$= L^{e_i} \times \quad \rightarrow \quad \boxed{k^{e_i} = \left(\frac{AE}{L}\right)^{e_i} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

2nd

$$f_r^e = \int \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} q(x)\,dx \approx r^e \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

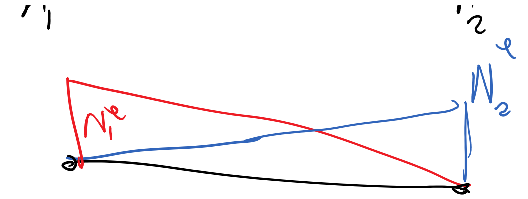
= if q is const or linear

$$r^e = \frac{L^e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$



$$\tilde{q}(x) = q_1 N_1^e(x) + q_2 N_2^e(x)$$

$$\underline{\tilde{q}(x)} = q_1 N_1^e(x) + q_2 N_2^e(x)$$



This function matches the end point values of q due to delta property of FE shape functions.

In fact, finite element shape functions are used to interpolate many things in FEM, etc.

solution

1) $u^h(x) = N_1(x) a_1 + N_2(x) a_2$

load

2) $q(x) \approx N_1(x) q_1 + N_2(x) q_2$

3) $x(\xi) = N_1(\xi) x_1 + N_2(\xi) x_2$

geometry

below

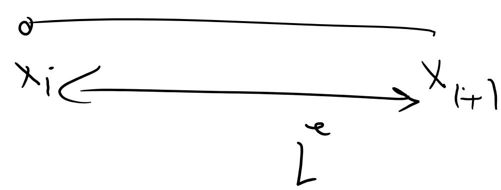
use $\tilde{q}$ instead of q

$$f_r^e = \int_e \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} \tilde{q}(x)\,dx = \int_e \left(\begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} [N_1^e \;\; N_2^e] \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \right) dx$$

take this out

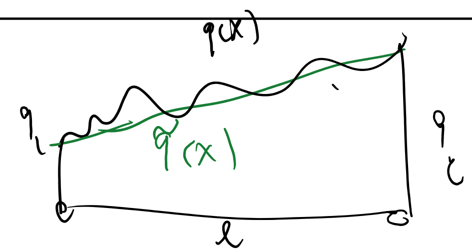
$$= \left(\int_e \begin{bmatrix} N_1^e \\ N_2^e \end{bmatrix} [N_1^e \;\; N_2^e]\, dx \right) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$= \underbrace{\left(\int_{x_i}^{x_{i+1}} \begin{pmatrix} \frac{x_{i+1}-x}{L^e} \\ \frac{x-x_i}{L^e} \end{pmatrix} \left[\frac{x_{i+1}-x}{L^e} \quad \frac{x-x_i}{L} \right] dx \right)}_{r^e} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$



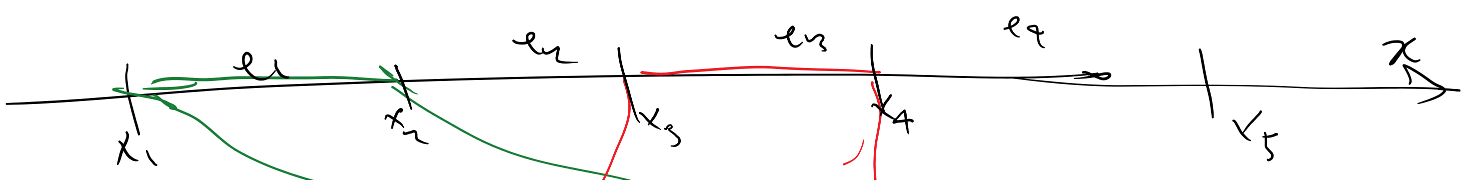
$$r^e = \frac{L^e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

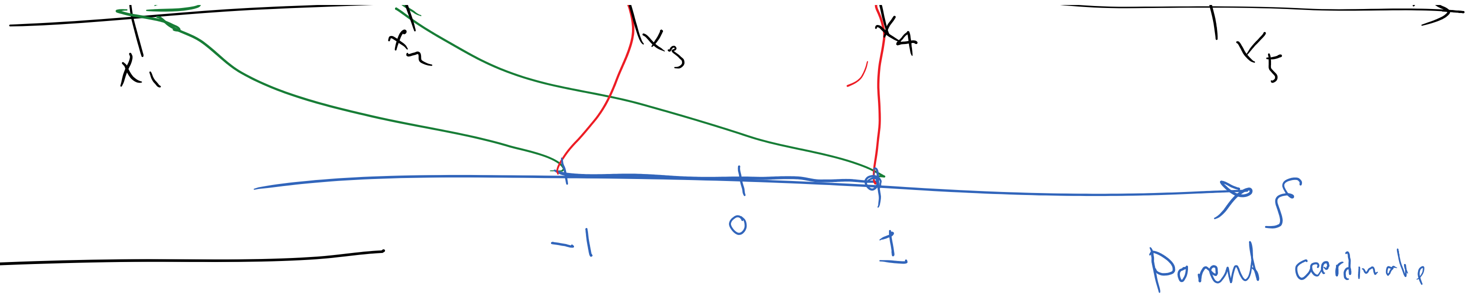
$$f_r^e = \int N^t q\, dx \approx r^e \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

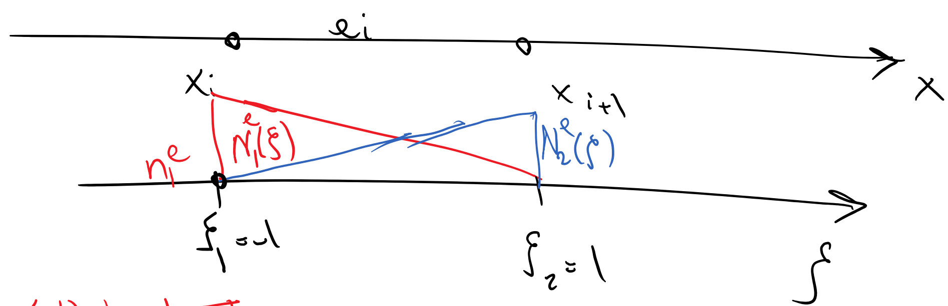


In FEM (and other numerical methods) we have discretization error (infinite unknowns to a finite number).
This introduces discretization error (FEM as Ch^2 for linear bar element).
As long as all the other errors go zero as fast or faster than discretization error, we are fine with them because eventually as h (element size) goes to zero, we converge to the exact solution

Last step to make all the elements be similar:







$$\boxed{\begin{array}{l} N_1(n_1)=1 \\ N_1(n_2)=0 \end{array}}$$

$N_1 = a\xi + b$

$N_1(n_1)=1 : \; N_1(\xi=-1) = a(-1)+b=1$

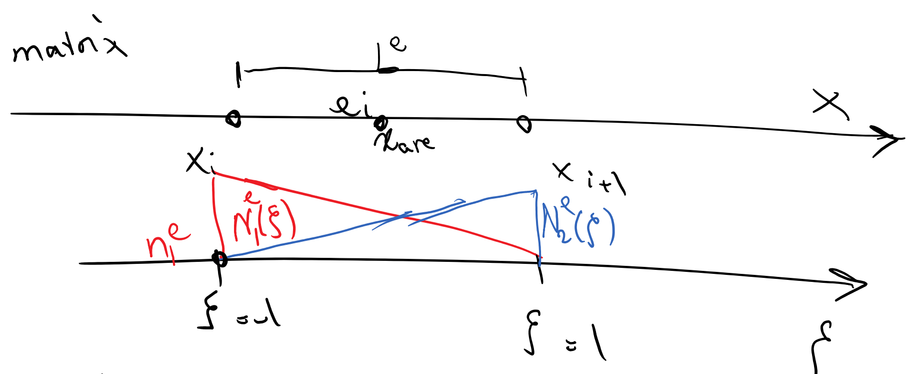
$N_2(n_2)=0 \quad N_1(\xi=1) = a(1)+b=0$ $\rightarrow$ $b=\frac{1}{2}$, $a=\frac{-1}{2}$ $\quad \boxed{N_1(\xi) = \frac{1-\xi}{2}}$

Similarly $\quad N_2(\xi) = \frac{1+\xi}{2}$

Shorter way $\quad N_1(\xi) = \frac{(\xi-\xi_2)}{(\xi_1-\xi_2)} = \frac{\xi-1}{-1-(1)} = \frac{1-\xi}{2}$

$$N_2(\xi) = \frac{\xi-\xi_1}{\xi_2-\xi_1} = \frac{\xi-(-1)}{1-(-1)} = \frac{\xi+1}{2}$$

Calculate the stiffness matrix



$$K^e = \int_e B^t D B \, dV$$

$$= \int_{x_i}^{x_{i+1}} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 \end{bmatrix} dx$$

1

(1) $B=?$ $\quad B = \frac{d}{dx}\begin{bmatrix} N_1 & N_2 \end{bmatrix}$, but $N_1(\xi) = \frac{1-\xi}{2}$ $\quad N_2 = \frac{1+\xi}{2}$

$B = \frac{d}{dx}\begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix}$ $\qquad x \longleftrightarrow \xi$

$x(\xi)$
we'll do it like this

$\xi=-1 \qquad x=x_i$

$\xi=1 \qquad x=x_{i+1}$

we'll do it like this $x(\xi)$

$\xi = 1 \qquad x = x_{i+1}$

$$x(\xi) = a + b\xi, \quad x(-1) = x_i, \quad x(+1) = x_{i+1}$$

$$a - b = x_i \qquad a + b = x_{i+1}$$

$$a = \frac{x_i + x_{i+1}}{2} = x_{ave} \qquad b = \frac{x_{i+1} - x_i}{2} = L_e/2$$

$$\boxed{x(\xi) = x_{ave} + \frac{L^e}{2}\xi}$$

Any easier way to relate ξ & x

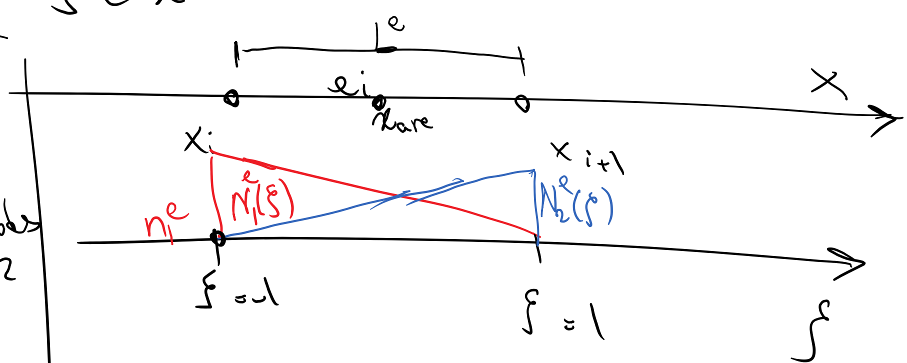
Hints

1) $u^e(\xi) = N_1(\xi)a_1 + N_2(\xi)a_2$

ξ dictions at nodes 1&2

2) $q(\xi) \approx N_1(\xi)q_1 + N_2(\xi)q_2$

source term



3) $x(\xi) = x_1^e N_1(\xi) + x_2^e N_2(\xi) = x_i\left(\frac{1-\xi}{2}\right) + x_{i+1}\left(\frac{1+\xi}{2}\right) =$

geometry

$x_1^e \to x_i \qquad x_2^e \to x_{i+1}$

$$= \frac{x_i + x_{i+1}}{2} + \left(\frac{x_{i+1} - x_i}{2}\right)\xi$$

$$= x_{ave} + \frac{L^e}{2}\xi$$

So, now we have a relation between x and $\xi \rightarrow$

Go back to the formula for the stiffness matrix

Back to equation 1:

$$K^e = \int_e B^t D B \, dV$$

$$= \int_{x_i}^{x_{i+1}} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 \end{bmatrix} dx$$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

will be expressed in terms of ξ

(1)

(1) $B = ?$ $\quad B = \frac{d}{dx}\begin{bmatrix} N_1 & N_2 \end{bmatrix}$, but $N_1(\xi) = \frac{1-\xi}{2} \quad N_2 = \frac{1+\xi}{2}$

$B = \frac{d}{dx}\left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2}\right]$

$x \longleftrightarrow \xi$

$x \Longleftrightarrow \xi$

$$\underline{B = \frac{d}{dx}\left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2}\right]}$$

$$B = \frac{d}{dx}\mathbf{N}(\xi) = \frac{dN(\xi)}{d\xi}\cdot\frac{d\xi}{dx} = \frac{dN(\xi)}{d\xi}\cdot\frac{1}{\underbrace{\frac{dx}{d\xi}}_{J}}$$

chain rule

$$x(\xi) = x_{ave} + \frac{L^e}{2}\xi$$

$$\boxed{J = \frac{dx}{d\xi} = \frac{L^e}{2}}$$

$$B = \frac{dN(\xi)}{d\xi}\cdot\frac{1}{J}$$

$$B = \frac{d}{dx}N$$

$$B_\xi = \frac{d}{d\xi}N$$

$$B = \frac{1}{J}B_\xi$$

$$B_\xi = \frac{d}{d\xi}\left[N_1(\xi) \quad N_2(\xi)\right] = \frac{d}{d\xi}\left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2}\right]$$

$$= \left[\frac{-1}{2} \quad \frac{1}{2}\right]$$

$$J = \frac{L^e}{2}$$

$$\Rightarrow \boxed{B = \left[-\frac{1}{L} \quad \frac{1}{L}\right]}$$

Again equation (1)

$$k^e = \int_e B^t D B \, dV$$

$$= \int_{x_i}^{x_{i+1}} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} EA \begin{bmatrix} B_1 & B_2 \end{bmatrix} dx$$

$\frac{1}{L}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}$

$$dx = \overbrace{\left(\frac{dx}{d\xi}\right)}^{J} d\xi \quad , \quad dx = J\, d\xi$$

(1)

(1) $B = ?$ $\quad B = \frac{d}{dx}\left[N_1 \quad N_2\right]$, but $N_1(\xi) = \frac{1-\xi}{2}$ $\quad N_2 = \frac{1+\xi}{2}$

$x \Longleftrightarrow \xi$

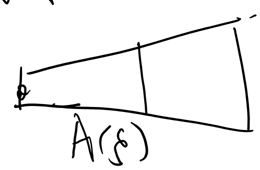
$$B = \frac{d}{dx}\left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2}\right]$$

$$k^e = \int^{1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} EA(\xi) \begin{bmatrix} -1 & 1 \end{bmatrix} \left(\frac{L^e}{2}\right) d\xi$$

$$k^e = \int_{\xi=-1}^{1} \begin{bmatrix}-1\\ 1\end{bmatrix} EA(\xi) \begin{bmatrix}-1 & 1\end{bmatrix} \left(\frac{\ }{2}\right) d\xi$$

FEM stiffness

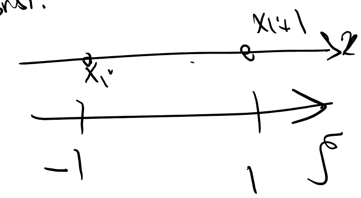
$$k^e = \frac{1}{2L^e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\int_{-1}^{1} EA(\xi)\, d\xi \leftarrow \text{general}$$

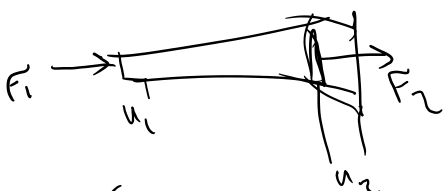


if $EA(\xi)$ is constant

AE const.

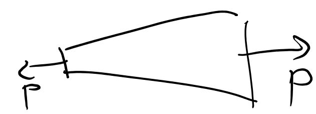
we recover $k^e = \left(\frac{AE}{L}\right)^e \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$





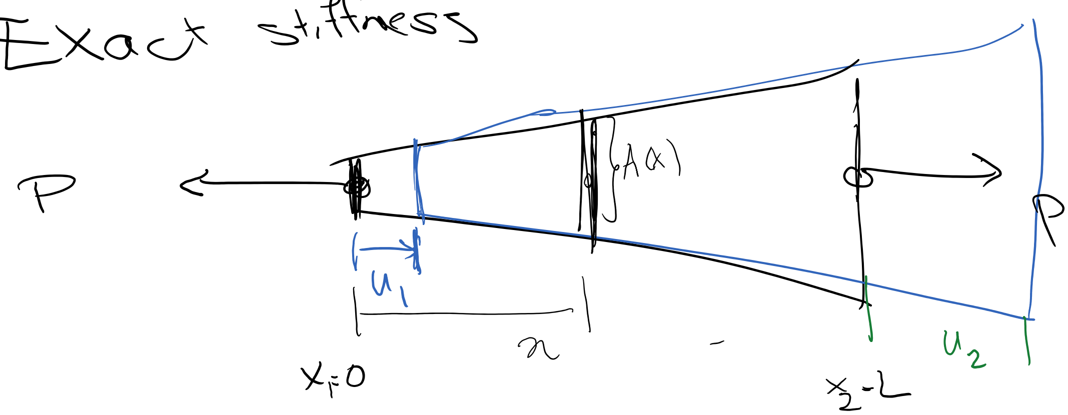
$\Delta u = u_2 - u_1$

$P = F_2 = -F_1$



$P = k^e_{11} \Delta u$

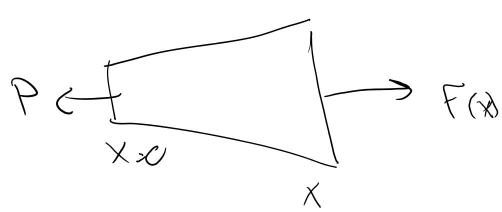
Exact stiffness



$$\varepsilon(x) = \frac{du(x)}{dx} \rightarrow u(x_2) - u(x_1) = \int_{x_1=0}^{x_2=L} \varepsilon(x)\, dx \qquad (i)$$

at position x

Force



$F(x) = P$ equilibrium

$\varepsilon(x)$ $\frac{P}{\ }$

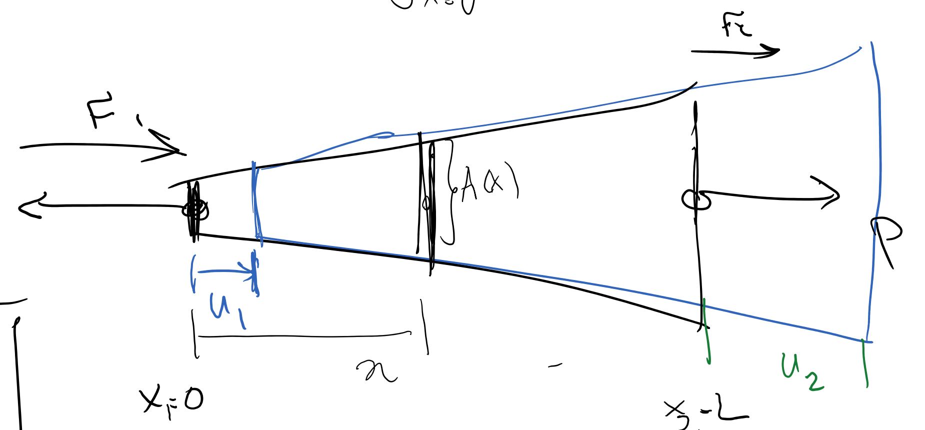
$F(x) = P$ equilibrium

$$\underbrace{\sigma(x)}_{\text{stress}} = \frac{F(x)}{A(x)} = \frac{P}{A(x)}$$

$$\varepsilon(x) = \frac{\sigma(x)}{E(x)} = \frac{P}{E(x)A(x)} = \frac{P}{EA(x)} \quad (ii)$$

Plug ii into (i) $\quad \Delta u = u_2 - u_1 = \int_{x=0}^{x=L} \frac{P\,dx}{AE(x)}$

$$\Delta u = u_2 - u_1 = P\int_{x=0}^{x=L} \frac{dx}{AE(x)}$$



$$\boxed{\frac{P}{\Delta u} = \frac{P}{u_2 - u_1} = K^{exact} = \frac{1}{\int_0^L \frac{dx}{AE(x)}}}$$

$$P = k^{exact}(u_2 - u_1)$$

$$F_2 = P = k^{exact}(u_2 - u_1)$$

$$F_1 = -P = -k^{exact}(u_2 - u_1)$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \underbrace{k^{exact}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{exact stiffness}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad k^{exact} = \frac{1}{\int_0^L \frac{dx}{AE(x)}}$$

k^{exact} for AE = constant $\qquad K^{exact} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ which matches FEM

In your HW for an example, you'll realize that FEM give you a stiffer solution (stiffness). FEM is always stiffer than real solution