2022/10/27

Thursday, October 27, 2022 9:55 AM

1D hyperbolic formulation

$$\int_e \hat{U}(\rho \dot{v} + d v - s)\, dV + \int_e \widehat{\nabla u}\, \sigma \, dV - \int_{\partial e} \hat{U} \sigma_n^{*}\, ds$$

$$- \tilde{\tau}_e \oint \hat{\sigma} \cdot n (v^{*} - v)\, ds = 0$$

$\hat{\sigma} = k \nabla \hat{u}$

$\alpha [\nabla u][v] = [U]^T k [\nabla u]$

Tk

choice for $\tilde{\tau} = \Delta T$ or $\frac{h_e}{c}$

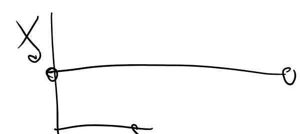
Deriving the matrices for 1D:

$$\int \hat{u} \rho \dot{v}\, dV = \underbrace{\left(\int U^T \rho U \, dV \right)}_{\text{goes to mass matrix}} \ddot{a}$$

$u = U(x) a(t) \qquad \dot{v} = U(x)\, \ddot{a}(t)$

U: basis functions

$$\boxed{M^e = \int_e U^t \rho U \, dV} \quad (1)$$



$\xi = \frac{x_e}{h}$

$U = [1 \quad x]$

$\nabla U = \frac{dU}{dx} = [0 \quad \frac{1}{h}]$

Eventual form $\quad M\ddot{a} + C\dot{a} + Ka = F$

M: mass, C: damping, K: stiffness

$$\int \hat{u} d v \, dV = \underbrace{\left(\int \hat{U} d U \, dV \right)}_{\text{contributes to } C \text{ matrix}} \dot{a}$$

$$\boxed{C^e_{body} = \int_e \hat{U} d U \, dV} \quad (2)$$

$$\int \nabla \hat{u} \sigma \, dV = \int \nabla \hat{u} (k \nabla u)\, dV$$

$\sigma = kg = k \nabla u$

$$\boxed{K^e = \int \nabla \hat{U} k \nabla U \, dV} \quad (3)$$

$$\boxed{K_b^e = \int \nabla \hat{U} k \nabla V \, dV} \quad (3)$$

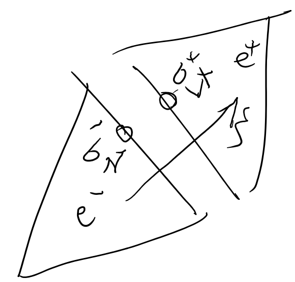
RHS

$$\int \hat{U} s \, dV \longrightarrow \boxed{F_b = \int_e \hat{U} s \, dV} \quad (4)$$

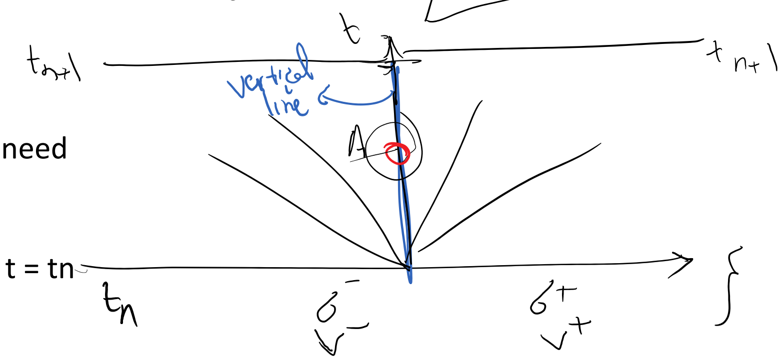
Boundary terms:

$$\int_{\partial e} \left[\hat{U} \sigma_n^* - \varepsilon \tilde{f} \hat{\alpha}_n (V^* - V) \right] ds$$

Choices of * values   average $V^* = \frac{V^- + V^+}{2}$, $\quad \sigma^* = \frac{\sigma^- + \sigma^+}{2}$



or we can use Riemann solution

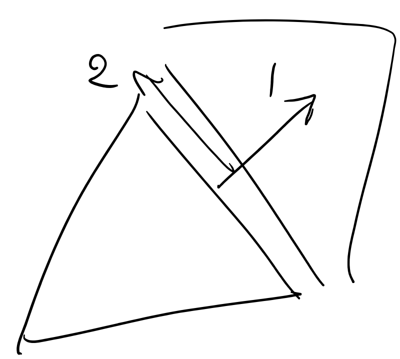


- For time marching schemes (space DG) we only need the solution on the vertical line.
- If we have source term, we need to compute its contribution to the star value for the finite time t = tn + Delta t /2

Sample Riemann solution (this problem):

$$\begin{cases} \rho \dot{V} + dV - \sigma_{,x} = s \\ \dot{\sigma} - k V_{,x} = 0 \end{cases}$$

PDE



1D

$$\sigma = k \nabla u = k u_{,x} \longrightarrow \dot{\sigma} = k V_{,x}$$

$$\begin{cases} \dot{V} - \frac{1}{\rho} \sigma_{,x} = -\frac{d}{\rho} V + s \\ \dot{\sigma} - k V_{,x} = 0 \end{cases}$$

$$\left\{ \dot{\delta} - k\, v_{,x} = 0 \right.$$

$$q = \begin{bmatrix} v \\ \delta \end{bmatrix} \qquad \dot{q} + A\, q_{,x} = S$$

primary variables

$$A = \begin{bmatrix} 0 & -\frac{1}{\rho} \\ -k & 0 \end{bmatrix}, \quad S = \begin{bmatrix} -\frac{d}{\rho} v + s \\ 0 \end{bmatrix}$$

Left eigen values

$$\ell = [\ell_1 \;\; \ell_2] \qquad [\ell_1 \;\; \ell_2] A = \lambda [\ell_1 \;\; \ell_2]$$

eigen vector #1 — left eigen vector

$$\underbrace{\begin{bmatrix} \ell^{(1)} \\ \hline \ell^{(2)} \end{bmatrix}}_{L} A = \underbrace{\begin{bmatrix} \lambda^1 & \\ & \lambda^2 \end{bmatrix}}_{\Lambda} \begin{bmatrix} \ell^{(1)} \\ \hline \ell^{(2)} \end{bmatrix} \qquad LA = \Lambda L$$

$$\dot{q} + A q_{,x} = S$$

premultiply by L

$$\dot{(Lq)} + (LA)\, q_{,x} = LS$$

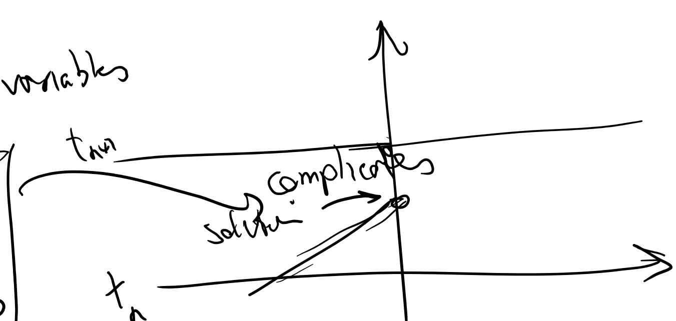
$$\dot{(Lq)} + \Lambda L q_{,x} = \text{"}$$

$$\dot{(Lq)} + \Lambda (Lq)_{,x} = LS$$

characteristic $\leftarrow \omega$

$\omega = Lq$ characteristics variables

$$\boxed{\dot{\omega} + \Lambda\, \omega_{,x} = S_\omega \qquad S_\omega = LS}$$



ignore S_ω for now

for $A = \begin{bmatrix} 0 & -\frac{1}{\rho} \\ k & 0 \end{bmatrix}$ $\quad L = \begin{bmatrix} c\rho & \\ -\rho & \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} -c & 0 \\ & \end{bmatrix}$

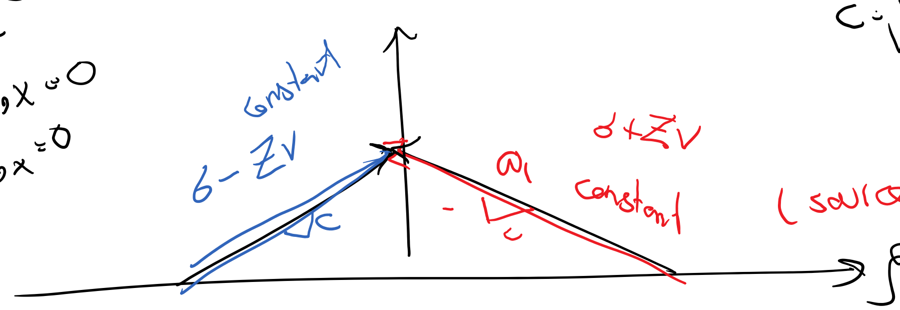
for $A = \begin{bmatrix} 0 & -\frac{1}{\rho} \\ -k & 0 \end{bmatrix}$ $L = \begin{bmatrix} c\rho & 1 \\ -c\rho & 1 \end{bmatrix}$ $\Lambda = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix}$

$c = \sqrt{\frac{k}{\rho}}$ wave speed $Z = c\rho$ impedance $\rho\ddot{u} + d\dot{u} - \nabla\cdot k\nabla u = -$

$c = \sqrt{\frac{k}{\rho}}$

$\dot{\omega}_1 + \lambda_1 \omega_{1,x} = 0$, $\lambda_1 = -c$

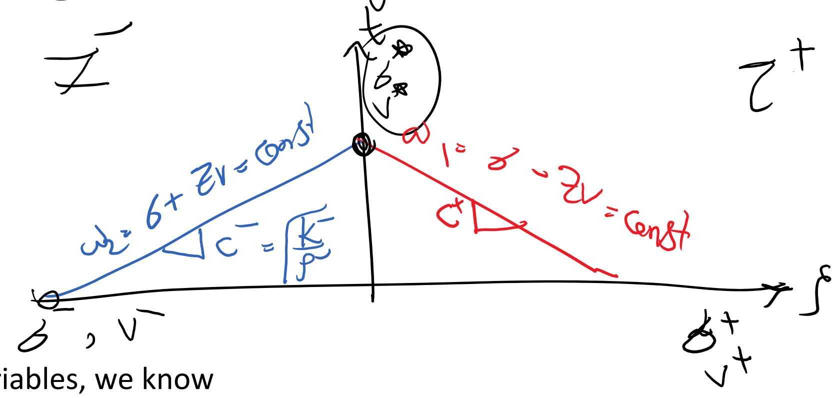
$\dot{\omega}_2 + \lambda_2 \omega_{2,x} = 0$, $\lambda_2 = c$



(source term we'd have an ODE here)

$$\omega = Lq = \begin{bmatrix} Z & 1 \\ -Z & 1 \end{bmatrix} \begin{bmatrix} v \\ \sigma \end{bmatrix} = \begin{bmatrix} \sigma + Zv \\ \sigma - Zv \end{bmatrix} \begin{matrix} \omega_1 \\ \omega_2 \end{matrix}$$

Solution for different Z on left & right



By choosing v and sigma as primary variables, we know that they are continuous across the vertical interface for general boundary (as opposites to g = grad u and p = rho v)

$$\begin{cases} \sigma^* - Z^- v^* = \sigma^- - Z^- v^- \\ \sigma^* + Z^+ v^* = \sigma^+ + Z^+ v^+ \end{cases} \longrightarrow$$

$$\boxed{\begin{aligned} \sigma^* &= \left(\frac{Z^- \sigma^+ + Z^+ \sigma^-}{Z^- + Z^+} \right) + \frac{Z^- Z^+}{Z^- + Z^+} (v^+ - v^-) \\ v^* &= \frac{1}{Z^- + Z^+} (\sigma^+ - \sigma^-) + \frac{Z^- v^- + Z^+ v^+}{\ldots} \end{aligned}}$$

$$v^* = \frac{1}{Z^- + Z^+}\left(\sigma^+ - \sigma^-\right) + \underbrace{\frac{Z^- v^- + Z^+ v^+}{Z^- + Z^+}}_{\text{average flux}}$$

if $Z^- = Z^+$

$$\sigma^* = \left(\overbrace{\frac{\sigma^- + \sigma^+}{2}}^{\text{average flux}}\right) + \frac{Z}{2}\left(v^+ - v^-\right)$$

$$v^* = \frac{1}{2Z}\left(\sigma^+ - \sigma^-\right) + \underbrace{\left(\frac{v^- + v^+}{2}\right)}_{\text{average flux}}$$

Riemann solution: has two parts:

1. Weighted average of the quantity -> average for the same material case.
2. Jump of other field(s) -> results in energy dissipation in numerical setting

General expression of the * solutions for this LINEAR PDE:

$$\sigma^* = \Sigma_{\sigma^-}\sigma^- + \Sigma_{\sigma^+}\sigma^+ + \Sigma_{v^-} v^- + \Sigma_{v^+} v^+$$

$$v^* = V_{\sigma^-}\sigma^- + V_{\sigma^+}\sigma^+ + V_{v^-} v^- + V_{v^+} v^+$$

(★)

Average $\Sigma_{\sigma^-} = \Sigma_{\sigma^+} = V_{v^-} = V_{v^+} = \frac{1}{2}$, others $= 0$

Rieman $\Sigma_{\sigma^-} = \frac{Z^+}{Z^- + Z^+}$ $\Sigma_{\sigma^+} = \frac{Z^-}{Z^- + Z^+}$ $\Sigma_{v^-} = \frac{-Z^- Z^+}{Z^- + Z^+}$ $\Sigma_{v^+} = -\Sigma_{v^-}$

$V_{\sigma^-} = \frac{-1}{Z^- + Z^+}$, $V_{\sigma^+} = -V_{\sigma^-}$, $V_{v^-} = \frac{Z^-}{Z^- + Z^+}$ $V_{v^+} = \frac{Z^+}{Z^- + Z^+}$

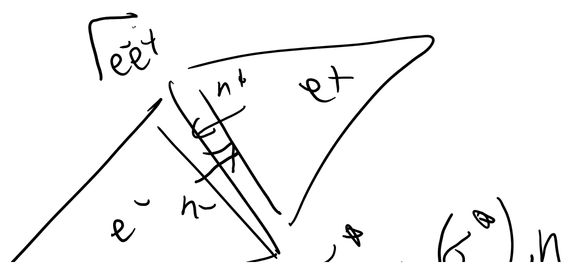
Let's now simplify the element boundary terms

$$\boxed{\int_{\partial e} \left[-\hat{v}\,\sigma^*_n - \varepsilon \tilde{T}\,\hat{\sigma}_n\left(v^* - v\right)\right] ds} \quad (★★)$$

Case 1 interior interface

from $-$

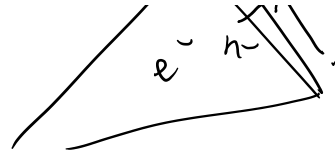
$$B = \int \hat{v}^-\left(-\sigma^*\cdot n^-\right) - \varepsilon \tilde{T}\,\hat{\sigma}^-\cdot n^-\left(v^* - v^-\right)$$



$$B_\Gamma = \int \hat{U}^- (-\sigma^* \cdot n^-) - \varepsilon \tilde{T} \sigma \cdot n (V^* - V^-)$$

$$+ \int_{e^+} \hat{U}^+ (-\sigma^* \cdot n^+) - \varepsilon \tilde{T} \hat{\sigma}^+ \cdot n^+ (V^* - V^+)$$

from + side



$\sigma^*_n = (\sigma^*) \cdot n$ for each side

$$B_\Gamma(\hat{U}, U) = \int -\underbrace{(\hat{U}^- n^- + \hat{U}^+ n^+)}_{[\hat{U}]} \cdot \sigma^*$$

$$- \varepsilon \tilde{T} \Big\{ \underbrace{(\hat{\sigma}^- n^- + \hat{\sigma}^+ n^+)}_{[\hat{\sigma}]} V^* + \varepsilon \tilde{T} \underbrace{(\hat{\sigma}^- \cdot n^- V^- + \hat{\sigma}^+ n^+ V^+)}_{\text{need to simplify this}} \Big\}$$

$$\sigma^- \cdot n^- \left(\frac{V^- + V^+}{2} + \frac{V^- - V^+}{2} \right) + \sigma^+ n^+ \left(\frac{V^- + V^+}{2} + \frac{V^+ - V^-}{2} \right)$$

as I did this before we'll get jumps & averages