2022/11/01

Tuesday, November 1, 2022 9:36 AM

From last time

$$B_\Gamma(\hat{U},U) = \int -\left(\hat{U}^- n^- + \hat{U}^+ n^+\right)\cdot \sigma^*$$

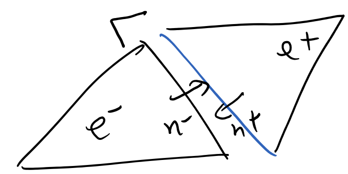
$$= \varepsilon \tilde{T}\Big\{ \underbrace{\left(\hat{\sigma}^- n^- + \hat{\sigma}^+ n^+\right)}_{[\hat{\sigma}]} v^* + \varepsilon \tilde{T} \underbrace{\left(\hat{\sigma}\cdot n^- v^- + \sigma^+ n^+ v^+\right)}_{\text{need to simplify this}} \Big\}$$

$$\sigma^- \cdot n^- \left(\frac{v^- + v^+}{2} + \frac{v^- - v^+}{2}\right) + \sigma^+ n^+ \left(\frac{v^+ + v^-}{2} + \frac{v^+ - v^-}{2}\right)$$

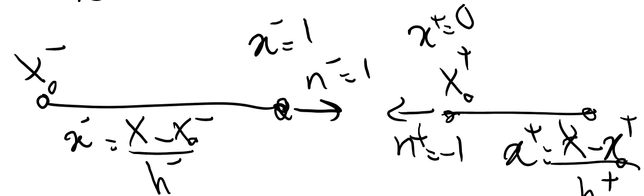
as I did this before we'll get jumps & averages

$$B_\Gamma(\hat{U},U) = -[U]\sigma^*$$

$$-\varepsilon\tilde{T}[\hat{\sigma}]v^* + \varepsilon\tilde{T}\left(\hat{\sigma} n^- v^- + \sigma^+ n^+ v^+\right)$$



1D



1D $[\phi] = \phi^- - \phi^+$

$$u^- = [1 \; x^-]\begin{bmatrix} a_1^- \\ a_2^- \end{bmatrix} \qquad u^+ = [1 \; x^+]\begin{bmatrix} a_1^+ \\ a_2^+ \end{bmatrix}$$

$$u^- = [1 \;\; 1]\, a^- \qquad u^+ = [1 \;\; 0]\, a^+$$

$$v^- = [1 \;\; 1]\, \dot{a}^- \qquad v^+ = [1 \;\; 0]\, \dot{a}^+$$

$$\sigma^- = k^- u_{,X} = [0 \;\; \tfrac{k^-}{h^-}]\, a^- \qquad \sigma^+ = [0 \;\; (\tfrac{k}{h})^+]$$

$$[u] = u^- - u^+ = [1 \;\; 1 \;\; -1 \;\; 0]$$

$$U = [1 \;\; 1 \;\; 1 \;\; 0]\, a$$

$$V = \dot{u} = [1 \; 1 \; 1 \; 0]\, \dot{a}$$

$$-[\hat{u}]\sigma^*$$

$$\sigma^* = \Sigma_{\sigma^-}\sigma^- + \Sigma_{\sigma^+}\sigma^+ + \Sigma_{v^-} v^- + \Sigma_{v^+} v^+$$

$$J_u = -[\hat{u}]\sigma^* = -\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}\Big(\Sigma_{\sigma^-}[0 \;\; \tfrac{k^-}{h^-} \;\; 0 \;\; 0]\, a + \Sigma_{\sigma^+}[0 \;\; 0 \;\; \tfrac{k^+}{h^+}]\, a$$

$$+ \underbrace{\Sigma_{v^-}[1 \;\; 1 \;\; 0 \;\; 0]\, \dot{a} + \Sigma_{v^+}[0 \;\; 0 \;\; 1 \;\; 0]\, \dot{a}}\Big)$$

→ these terms go to damping matrix

– Riemann solution Σ_{V^-} & $\Sigma_{V^+} \neq 0$ Riemann solution introduces some damping

$$\boxed{\begin{aligned} J_u &= K_1^\Gamma a + C_1^\Gamma \dot{a} \\ K_1^\Gamma &= \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \left(\Sigma_{\delta^-} \left[0 \ \ \frac{k^-}{h^-} \ \ 0 \ \ 0 \right] + \Sigma_{\delta^+} \left[0 \ \ 0 \ \ 0 \ \left(\frac{k}{h}\right)^+ \right] \right) \\ C_1^\Gamma &= \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \left(\Sigma_{V^-} [1 \ 1 \ 0 \ 0] + \Sigma_{V^+} [0 \ 0 \ 1 \ 0] \right) \end{aligned}}$$

(4)

$$J_\delta = -\varepsilon \tilde{T} [\delta]^T v^* + \varepsilon \tilde{T}^* \left(\hat{\delta}^- n^- v^- + \hat{\delta}^+ n^+ v^+ \right)$$

$$= -\varepsilon \tilde{T} \begin{bmatrix} 0 \\ \left(\frac{k}{h}\right)^- \\ 0 \\ -\left(\frac{k}{h}\right)^+ \end{bmatrix} \Big(\underbrace{V_{\delta^-} \left[0 \ \ \frac{k}{h}^- \ \ 0 \ \ 0 \right] a + V_{\delta^+} \left[0 \ \ 0 \ \ 0 \ \left(\frac{k}{h}\right)^+ \right] a + V_{V^-} [1 \ 1 \ 0 \ 0] \dot{a} + V_{V^+} [0 \ 0 \ 1 \ 0] \dot{a}}_{\text{for both average \& Riemann we're adding these damping terms}} \Big)$$

$$\varepsilon \tilde{T} \begin{bmatrix} 0 \\ \left(\frac{k}{h}\right)^- \\ 0 \\ 0 \end{bmatrix} (1) [1 \ 1 \ 0 \ 0] \dot{a} + \varepsilon \tilde{T} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \left(\frac{k}{h}\right)^+ \end{bmatrix} (1) [0 \ 0 \ 1 \ 0] \dot{a}$$

(2)

$$I_\sigma = K_2 a + C_2 \dot{a}$$

$$K_2 = -\varepsilon \tilde{T} \begin{bmatrix} 0 \\ (k/h)^- \\ 0 \\ (k/h)^+ \end{bmatrix} \begin{bmatrix} 0 & V_\sigma^- (k/h)^- & 0 & V_{\sigma^+} (k/h)^+ \end{bmatrix}$$

$$C_2 = \; // \; \Big(V_v^- \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} + V_v^+ \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$
$$+ \varepsilon \tilde{T} \begin{bmatrix} 0 \\ (k/h)^- \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} - \varepsilon \tilde{T} \begin{bmatrix} 0 \\ 0 \\ 0 \\ (k/h)^+ \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \Big)$$

$$K^\Gamma = K_1^\Gamma + K_2^\Gamma \qquad C^\Gamma = C_1^\Gamma + C_2^\Gamma$$

As we can see the element jumps introduce some damping terms to the system!
So, the DG formulation naturally dissipates energy even for lossless domains (damping coefficient = 0)

Essential BC:

$$B(\hat{u}, u) = -\int \hat{u} (\sigma^* \cdot n) \, ds - \varepsilon \tilde{T} \int \hat{\sigma} \cdot n (v^* - v) \, ds$$

$$v^* \to \bar{v}$$

$$B(\hat{u}, u) = -\hat{u} \underbrace{(\sigma \cdot n)}_{\text{stiffness}} - \varepsilon \tilde{T} \hat{\sigma} \cdot n (\underbrace{v^*}_{\text{RHS}} - \underbrace{v}_{\text{Damping}})$$

L: $\hat{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $n = -1$, $\sigma = \begin{bmatrix} 0 \\ k/h \end{bmatrix}$

R: $\hat{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $n = 1$, $\sigma = \begin{bmatrix} 0 \\ k/h \end{bmatrix}$

$-\hat{u}(\sigma \cdot n)$ $\quad -\hat{u}(\sigma \cdot n)$

$\to K^{\sigma D u} =$ $\quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{k}{h} \end{bmatrix}$ $\quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{k}{h} \end{bmatrix}$

$$K^{\partial D_0} = \qquad \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & \frac{k}{h}\end{bmatrix} \qquad \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0 & \frac{k}{h}\end{bmatrix}$$

$$C^{\partial D_u} \quad \varepsilon\tilde{T}\tilde{\sigma}\cdot n_V \qquad -\varepsilon\tilde{T}\begin{bmatrix}0\\ \frac{k}{h}\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} \qquad +\varepsilon\tilde{T}\begin{bmatrix}0\\ \frac{k}{h}\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}$$

$$F^{\partial D_u} \quad \varepsilon\tilde{T}\tilde{\sigma}\cdot n\bar{V} \qquad -\varepsilon\tilde{T}\begin{bmatrix}0\\ \frac{k}{h}\end{bmatrix}\bar{V} \qquad \varepsilon\tilde{T}\begin{bmatrix}0\\ \frac{k}{h}\end{bmatrix}\bar{V}$$

Natural BC:

$$B(\tilde{u},u) = -\int \tilde{u}\,\sigma_n^*\,ds - \varepsilon\tilde{T}\int \tilde{\sigma}\cdot n(V^* - V)\,ds$$

$$V^* = V \qquad \sigma_n^* \overset{\text{def}}{=} \bar{\sigma}_n$$

$$B(\hat{u},u) = - \quad \hat{u}\,\bar{\sigma}_n$$

$$F^{\partial D_f} \qquad \overset{L}{\circ}\!\!-\!\!-\!\!-\qquad -\!\!-\!\!-\overset{R}{\circ}$$

$$\begin{bmatrix}1\\0\end{bmatrix}\bar{\sigma}_n \qquad \begin{bmatrix}1\\1\end{bmatrix}\bar{\sigma}_n$$

as we can see because of interior faces C matrix is not block diagonal ☹

we cannot solve one element at a time

$$\underset{\text{block diagonal}}{M}\ddot{a} + C\dot{a} + Ka = F$$

we need to solve it globally

$$M\left(\frac{a_{n+1} + a_{n-1} - 2a_n}{\Delta t^2}\right) + C\left(\frac{a_{n+1} - a_n}{\Delta t}\right) + K a_n = F_n$$

$$(M + C\Delta t)\, a_{n+1} = RHS$$

$$\underbrace{(M + C\Delta t)}_{\text{not block diagonal}} a_{n+1} = RHS$$

With 1F formulation, it's better to use an implicit solver as we don't lose anything

Next time, I'll discuss how to address this issue