2022/11/03

Thursday, November 3, 2022 9:49 AM

$u, v, \frac{g = \nabla u}{\sigma}$, $\frac{\nabla u + \nabla u^T}{2}$ strain

$$\int_e \hat{v}(\rho \dot{v} + dv - \nabla \cdot \sigma - s)\, dV - \int_{\partial e} \hat{v}(\sigma_n^* - \sigma \cdot n)\, ds$$

$$-\varepsilon \left(\int_{\partial e} \hat{\sigma} \cdot n (v^* - v)\, ds + \int_e \hat{\sigma} : \cancel{(C^{-1} \dot{\sigma} - \nabla v)} \right)$$

$$\dot{g} - \tilde{\nabla} v$$

$$+ \alpha \int_e \hat{u}(\dot{u} - v) = 0$$

add a β here

$\sigma = C g \rightarrow g = C^{-1} \sigma$

2 field formulation

$\sigma = C g(u) = \begin{cases} C \nabla u & \text{wave} \\ C\left(\frac{\nabla u + \nabla u^T}{2}\right) & \text{solid mechanics} \end{cases}$

u, v

$$\boxed{\int_e \hat{v} \rho \dot{v}\, dV + \overbrace{\frac{1}{\tilde{T}^2} \int_e \hat{u} \rho \dot{u}\, dV}^{\alpha \text{ mass matrix}} + \underbrace{\int_e (\hat{v} dv + \nabla \hat{v}\, \sigma)\, dV}_{\rightarrow f(u)} - \underbrace{\int_{\partial e} (\hat{v} \sigma_n^* - \varepsilon \hat{\sigma} \cdot n (v^* - v))\, ds}_{\rightarrow f(u)} = \int_e \hat{v}\, s\, dV}$$

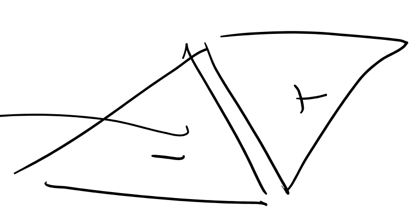
$\tilde{T} = \frac{h}{c}$ $\alpha = \left(\frac{h}{c}\right)^2$

$\sigma_n = \sigma^* \cdot n_e$ v^* now depend on $\sigma^\mp = C \nabla u^\mp$ & $v^\mp$

it does not involve any $\dot{u}$

$\bar{u}(x) = \bar{H}_u(x)\, U(t)$

$= [\bar{u}_1(x)\ \bar{u}_2(x) \ldots \bar{u}_n(x)] \begin{bmatrix} \bar{U}_1 \\ \bar{U}_2 \\ \vdots \\ \bar{U}_n \end{bmatrix}(t)$



$\bar{V}(x) = \bar{H}_v(x)\, V(t)$

$= [\bar{V}_1(x) \ldots \bar{V}_n(x)] \begin{bmatrix} \bar{V}_1 \\ \bar{V}_2 \\ \vdots \end{bmatrix}(t)$

$$= \left[V_1^-(x) \cdots V_n^-(x) \right] \begin{bmatrix} v_2^- \\ \vdots \\ v_n^- \end{bmatrix}(t)$$

$$a = \begin{bmatrix} U_1^-(t) \\ \vdots \\ U_n^-(t) \\ V_1^-(t) \\ \vdots \\ V_n^-(t) \end{bmatrix}$$

Since v is now interpolated, there is no contribution to damping matrix when dealing with star values $\Longrightarrow$ we'll end up with block diagonal matrix equations

$$M \dot{a} + \underline{f(a)} = F$$

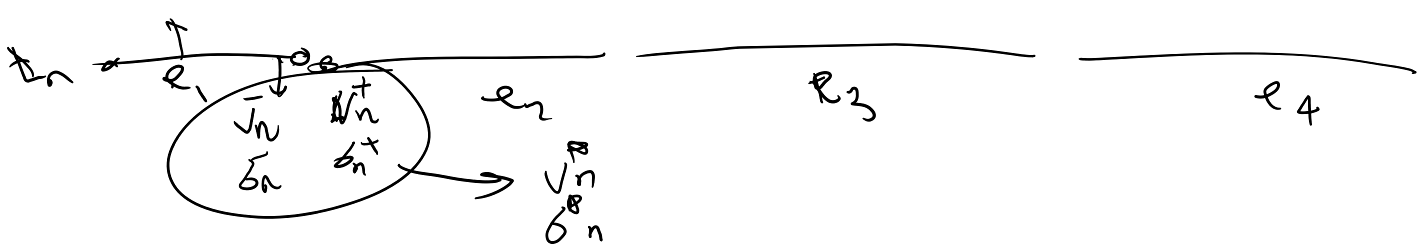
this linear in $f(a) = Ka$

Forward Euler $\quad M \left(\frac{a_{n+1} - a_n}{\Delta t} \right) + f(a_n) = F_n$

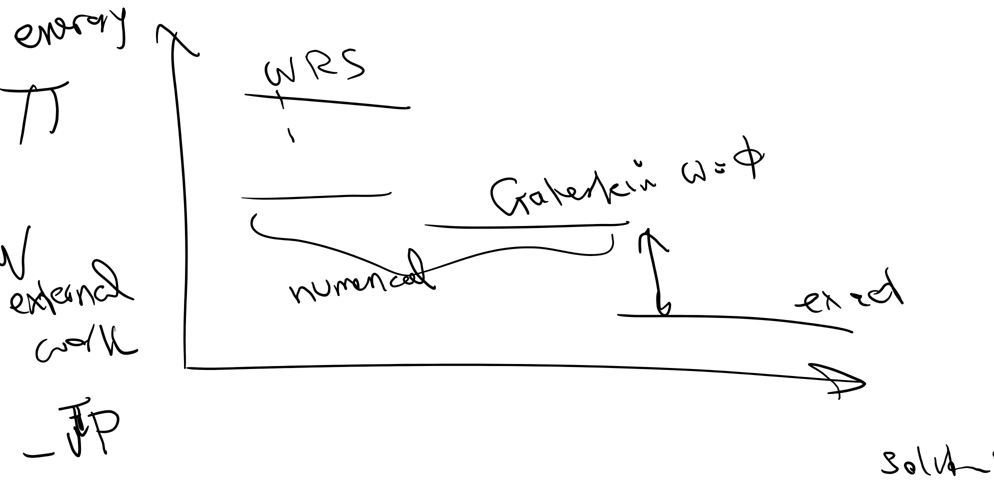
$$\underbrace{M}_{\text{block diagonal}} a_{n+1} = M a_n \cancel{+} \Delta t \underbrace{f(a_n)}_{\text{don't need to}} - F_n$$

we can solve it one element at a time

$M \dot{a} = -RHS \rightarrow$ all from time step n



side note



$$\Pi = \underset{\text{internal}}{U} - \underset{\text{external work}}{W}$$

$$= \underbrace{\tfrac{1}{2} u^T K u} - u^T P$$

Dynamic AC $\quad \rho \ddot{u} + d\dot{u} - \nabla \cdot \sigma = s \qquad s = \rho b$

steady state $(\dot{\ }) = 0 \qquad \boxed{-\nabla \cdot \sigma = s}$ Elliptic

fixed frequency $\quad u(x,t) = U(x)\, e^{i\omega t}$

$\sigma(x,t) = \Sigma(x) e^{i\omega t} \qquad s(x,t)$

$$\cancel{e^{i\omega t}}\left(\rho U(x)(i\omega)^2 + d\, U(x)(i\omega) - \nabla \cdot \Sigma(U)\right) = \cancel{e^{i\omega t}}\, S(x)$$

$$\boxed{-\rho U \omega^2 + i\omega d U - \nabla \cdot \Sigma = S(x)}$$

DC elliptic PDE

- Helmholtz equation is indeterminate (eigen values of stiffness are both positive & negative)
- Solution is highly

— Solution is highly

$$\boxed{\begin{aligned} U^* &= \{U\} \\ \Sigma^* &= \{\Sigma\} + \frac{\eta_0 C}{h}[U] \end{aligned}}$$

"these elliptic fluxes don't work well

$$\sigma^* = \{\sigma\} - \frac{Z}{2}(v^+ - v^-)$$

$$v^* = \{v\} - \frac{1}{2Z}(\sigma^+ - \sigma^-)$$

Riemann solution

$V = \dot{U}$ $\qquad V(x) = i\omega U(x)$

$$\boxed{\begin{aligned} \Sigma^* &= \{\Sigma\} - \frac{Z\, i\omega}{2}(U^+ - U^-) \\ U^* &= \{U\} - \frac{1}{2Z\, i\omega}(\Sigma^+ - \Sigma^-) \end{aligned}}$$

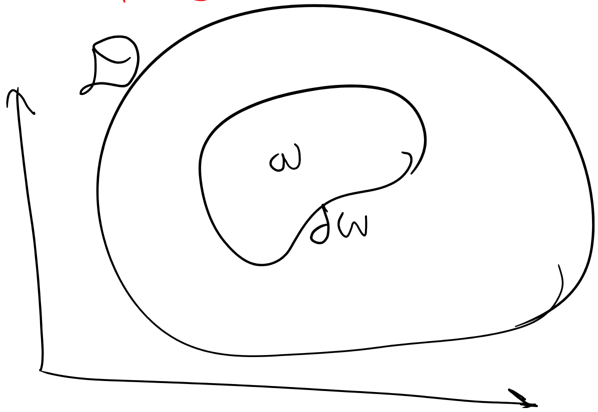
We can use these fluxes for Helmholtz equation

Spacetime balance laws

Steady state balance laws

$$-\int_{\partial\omega} f_x \cdot n \, ds + \int_{\omega} s \, dV = 0$$

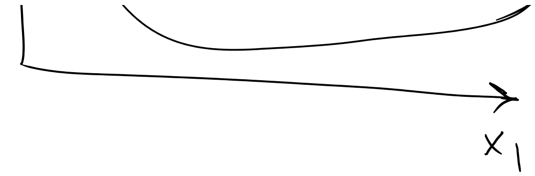
term



$$\int_{\partial\omega} \qquad \int_{\omega}$$

outward spatial flux density

source term



Examples:

Linear momentum $\quad f_x = -\sigma \quad s = \rho b \quad \int_{\partial\omega} \sigma \cdot n\,ds + \int_{\omega} \rho b\,dV = 0$

Energy (with only thermal effects) $\quad f_x = q \quad s = Q \quad -\int_{\partial\omega} q \cdot n\,ds + \int_{\omega} Q\,dV = 0$

Dynamic form

$$\frac{D}{Dt} \int_{\omega} f_t\,dV = -\int_{\partial\omega} f_x \cdot n\,ds + \int_{\omega} s\,dV$$

f_t: temporal flux of balanced quantity

only thermal effects $e = cT$

$$\frac{D}{Dt} \underbrace{\int e\,dV}_{E} = -\int_{\partial\omega} q \cdot n\,ds + \int Q\,dV$$

e: energy density per volume

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linear momentum

$$\frac{D}{Dt} P = \sum \text{forces}$$

$p = \rho\vec{v}$ = linear momentum density

$$P = \sum mV = \int_{\omega} \vec{v}\,\overbrace{(\rho\,dV)}^{dm} = \int_{\omega} \overbrace{(\rho\vec{v})}^{p}\,dV$$

$$\frac{D}{Dt} \underbrace{\int_{\omega} \vec{p}\,dV}_{P} = \underbrace{\int_{\partial\omega} \sigma \cdot n\,ds + \int_{\omega} \rho b\,dV}_{\sum F}$$

$p = \rho\vec{v}$

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have

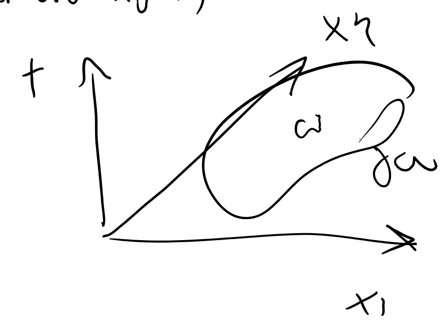
In general we have

$$\frac{D}{Dt}F = \frac{d}{dt}\int_{\omega} f_t \, dV = -\int_{\partial\omega} f_x \cdot n \, dS + \int_{\omega} s \, dV$$

Balanced quantity (P, E)

temporal flux density of balanced quantity

outward spatial flux density (1 tensor order higher)



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$e = U^h - U$

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$U^h \in S^h$

$a(w^h, e) = 0$kk

$a(e,e) \leq a(U^h - u, U^h - u) \rightarrow$

$a(w^h, u) = (w^h, f) + (w^h, h)_\Gamma$

$a(w^h, u^h) - a(w^h, u) = a(w^h, e)$

$0 = a(w^h, u^h)$

$a(e + w^h, e + w^h) = a(e,e) + 2a(\cancel{w^h}, e) + a(w^h, w^h)$

0

$a(w^h, w^h) \geq 0$

$a(e,e) \leq a(e + w^h, e + w^h)$

$U^h = u^h + w^h \quad , \; e = U^h - u$

$$\boxed{a(u,u) = a(u^h,u^h) + a(e,e)}$$

$$\hookrightarrow a(e,e) = a(u,u) - a(u^h,u^h)$$

$$\underbrace{a(u^h,u^h)}_{a(\cdot,\cdot)} \leq a(u,u)$$