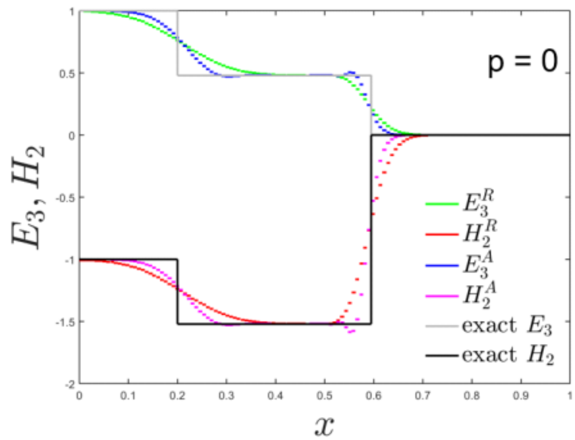
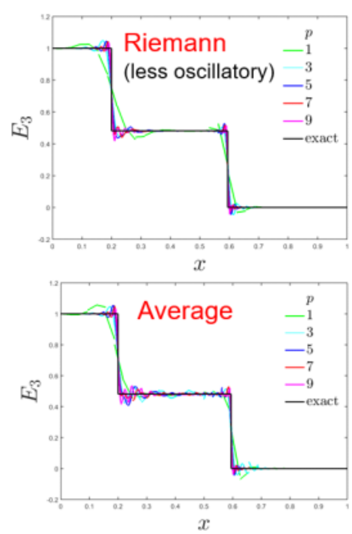
2022/11/08

Tuesday, November 8, 2022 9:53 AM

Some notes I forgot before:
Riemann solution is more dissipative



Riemann solutions: More dispersive
(especially for low p)



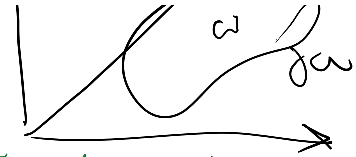
Continuing from the last time:

$$\frac{D}{Dt} F = \frac{d}{dt}\int_{\omega} f_t \, dV = -\int_{\partial\omega} f_x \cdot n \, dS + \int_{\omega} s \, dV$$

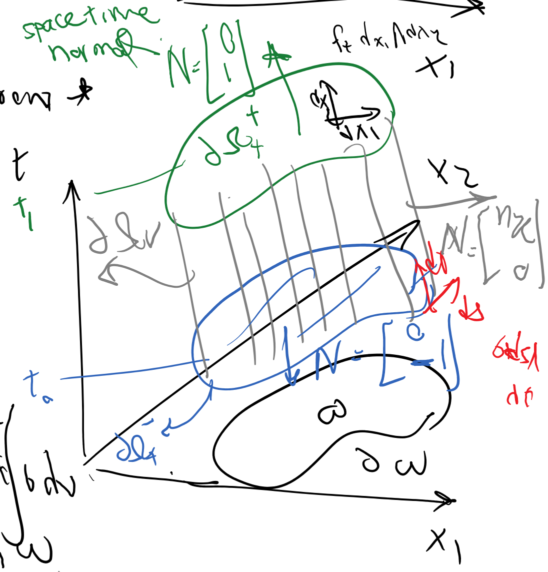
Balanced quantity (P, E)

temporal flux density of balanced quantity

outward spatial flux density (1 tensor order higher)



Deriving space time form of balance law from *



integrate from to to t_1

source term

$$\frac{d}{dt}\int_\omega f_t\, dV = -\int_{\partial\omega} f_x\cdot n_x\, dS + \int_\omega r\, dV$$

$$\int_\omega f_t\, dV\Big|_{t=t_1} - \int_\omega f_t\, dV\Big|_{t=t_0} = -\int_{t_0}^{t_1}\int_{\partial\omega} f_x\cdot n_x\, dS\, dt + \int_{t=t_0}^{t_1}\int_\omega r\, dV\, dt$$

$$\int_{\partial\Omega_t^+}\begin{bmatrix}f_x\\f_t\end{bmatrix}\cdot\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{N \text{ on } \partial\Omega_t^+} dS + \int_{\partial\Omega_t^-}\begin{bmatrix}f_x\\f_t\end{bmatrix}\underbrace{\begin{bmatrix}0\\-1\end{bmatrix}}_{N \text{ on } \partial\Omega_t^-} dS = -\int_{\partial\Omega_V}\begin{bmatrix}f_x\\f_t\end{bmatrix}\cdot\underbrace{\begin{bmatrix}n_x\\0\end{bmatrix}}_{N \text{ on } \partial\Omega_V} dS \qquad \Omega = \omega\times[t_0,t_1]$$

$$\int_\Omega r\, dV$$

$F=\begin{bmatrix}f_x\\f_t\end{bmatrix}$ spacetime flux density $\qquad N=\begin{bmatrix}\vec{n}_x\\n_t\end{bmatrix}$ spacetime normal

$$\int_{\partial\Omega_t^+} F\cdot N\, dS + \int_{\partial\Omega_t^-} F\cdot N\, dS + \int_{\partial\Omega_V} F\cdot N\, dS = \int_\Omega r\, dV$$

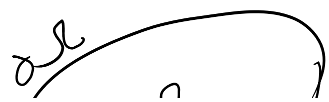
$F=\begin{bmatrix}f_x\\f_t\end{bmatrix}$

$$\oint_{\partial\Omega} F\cdot N\, dS = \int_\Omega r\, dV$$

$N=\begin{bmatrix}\vec{n}_x\\n_t\end{bmatrix}$

+

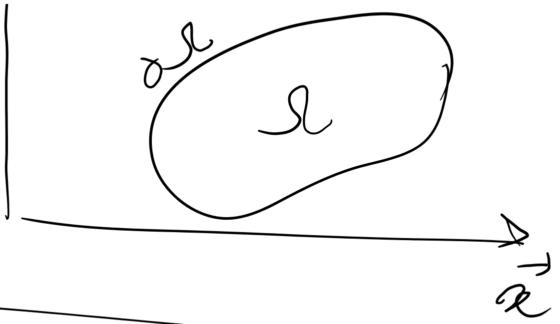
in fixed



$[\;]^t$

in fact it holds for any shape Ω

(**)



Although we derived (**) (spacetime form) from (*), (**) is a more general statement of balance laws and should be the starting point.
There is, however, a problem in (**) as we use the notation of normal vector in spacetime. The problem is that we don't have a metric in spacetime.

Remedies:
- Multiply the time axis by a scale of velocity (e.g. light speed for electromagnetics, etc.) so all axes have the unit of space …
- Language of differential forms: This is the common approach in relativity

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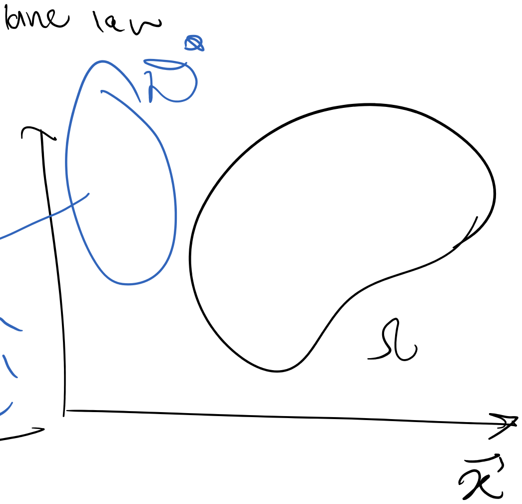
What's the next step after spacetime balance law

$$\forall \Omega \int_{\partial\Omega} F\cdot N\, dS = \int_{\Omega} r\, dV$$

Balance law t

for $\Omega \subseteq \mathring{D}$

the solution is smooth enough F to be C^1



$$\int_{\Omega} \nabla_{st}\cdot F\, dV = \int_{\Omega} r\, dV$$

$$\forall \Omega \subseteq \mathring{D} \int_{\Omega} (\nabla_{st}\cdot F - r)\, dV = 0$$

arbitrary

localization

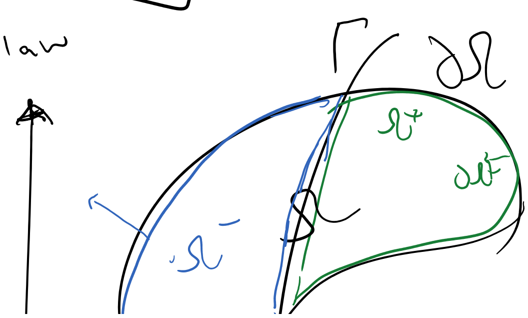
(4) $\boxed{\forall x \in \mathring{D} \quad \nabla_{st}\cdot F - r = 0}$

PDE

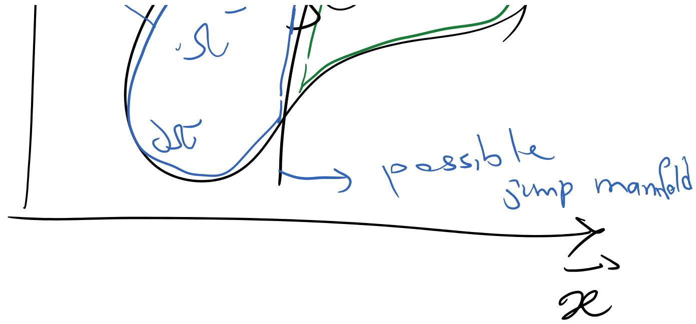
What else do we get from Balance law

(+) $\int_{\partial\Omega^-} F\cdot N\, ds = \int_{\Omega^-} r\, dV$

(−) $\int F\cdot N\, ds = \int r\, dV$



② $\int_{\partial\Omega^+} F\cdot N\,dS = \int_{\Omega^+} r\,dV$



③ $\int_{\partial\Omega} F\cdot N\,dS = \int_{\Omega} r\,dV$

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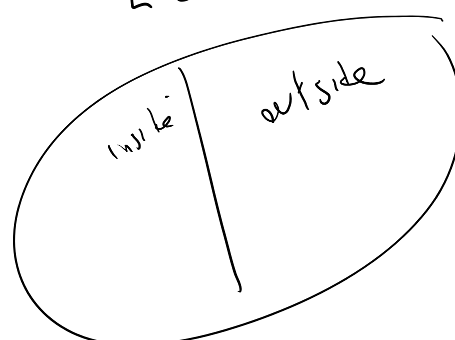
① - ② - ③

$$\forall\Omega \int_{\Gamma\cap\Omega} \left(F^-\cdot N^- + F^+\cdot N^+\right) ds = 0 \longrightarrow$$

localization

$$F^- N^- + F^+ N^+ = 0$$

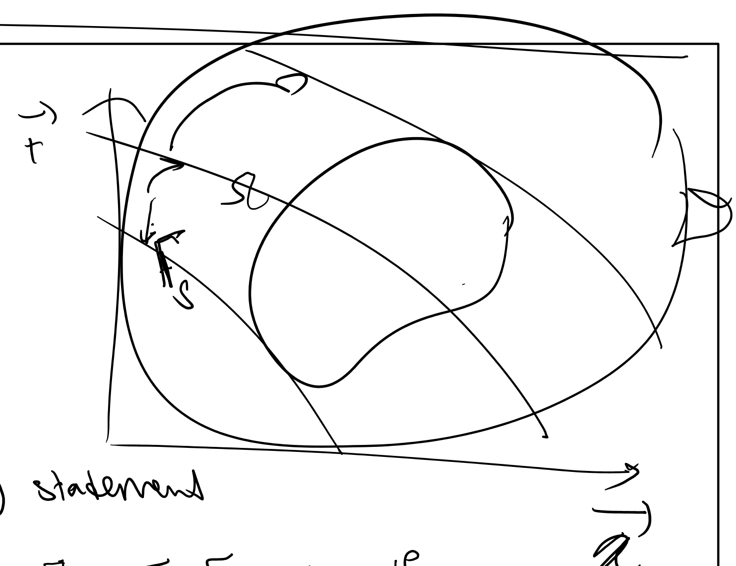
$$[F] = 0$$



$$[[F]] = F_{outside} - F_{inside} = F^+ - F^-$$

$$[[F]]\cdot N^- = 0$$

Summary



Balance law

$$\forall\Omega \quad \int_{\partial\Omega} F\cdot N\,dS = \int_\Omega r\,dV$$

$f_t\cdot n_t + f_x\cdot n_x$

strong statement

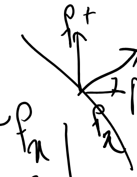
① PDE $\forall x\in D\setminus\Gamma$ $\quad \nabla_{st}\cdot F - r = 0$; $\frac{df_t}{dt} + \nabla_s\cdot f_x = r$

② Jump conditions $\quad [[F]]: F^- N^- + F^+ N^+ = 0$

② Jump conditions

$$\forall x \in D\Gamma_s \quad [F] = F^- N^- + F^+ N^+ = 0$$

$$(f_x^{\circ} - f_x) \cdot n_x + (f_t^{\circ} - f_t) n_t = 0$$



$$\begin{bmatrix} f_x \\ f_t \end{bmatrix} \quad \begin{bmatrix} f_x^{\circ} \\ f_t^{\circ} \end{bmatrix}$$

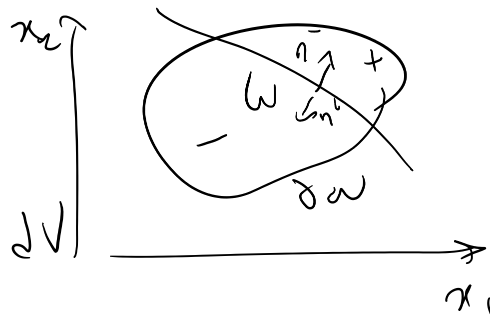
Also called Rankine Hugoniot jump condition

More discussion on the jump condition

Solid mechanics

$f_x = -\sigma$

$f_t = \rho V = P$



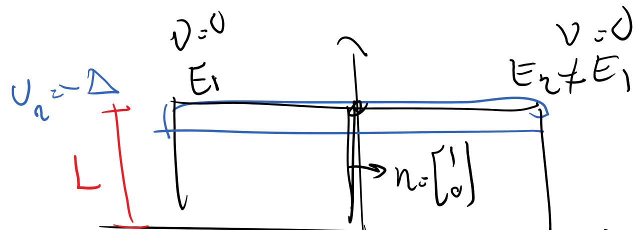
$$\int_{\partial\omega} f_x \cdot n_x \, ds = \int_{\omega} r \, dV$$

$$\rightarrow \quad [f_x] = f_x^- n_x^- + f_x^+ n_x^+ = 0$$

$$-\left(\sigma^- \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sigma^+ \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$t^- - t^+ = 0$$

$\sigma^- \neq \sigma^+$ in general

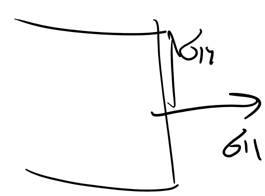


only these are equal

$$\begin{bmatrix} \sigma_{11}^- & \sigma_{12}^- \\ \sigma_{12}^- & \sigma_{22}^- \end{bmatrix} \quad \begin{bmatrix} \sigma_{11}^+ & \sigma_{12}^+ \\ \sigma_{12}^+ & \sigma_{22}^+ \end{bmatrix}$$

$$\sigma_{22}^- = \frac{-\Delta}{L} E_1 \qquad \sigma_{22}^+ = \frac{-\Delta}{L} E_2$$

$$\sigma_{22}^- \neq \sigma_{22}^+$$



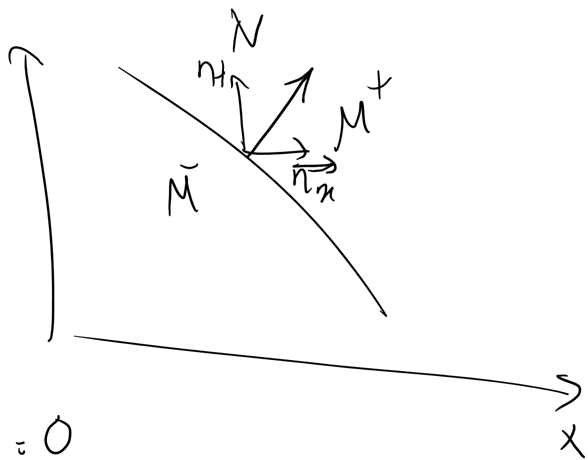
$$F = \begin{bmatrix} f_x \\ f_t \end{bmatrix} \qquad [F] = F^- N^- + F^+ N^+ = 0$$

$$F = \begin{pmatrix} \tau_x \\ f_t \end{pmatrix} \qquad [F] = F^- N^- + F^+ N^+ = 0$$

$$(F^+ - F^-) \cdot N^- = 0$$

But $F^+ - F^-$ is not generally zero

Jump conditions are more interesting in dynamics



Elastodynamics

$$[\![g]\!] = g^+ - g^-$$

$$M = \begin{bmatrix} -\sigma \\ p \end{bmatrix}$$

spacetime linear momentum density

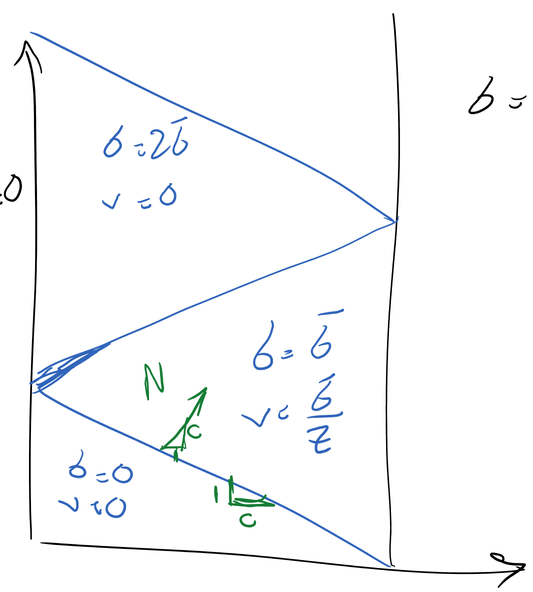
$$\begin{bmatrix} [\![-\sigma]\!] \\ [\![p]\!] \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_t \end{bmatrix} = 0$$

$$\boxed{-[\![\sigma]\!]\, n_x + [\![p]\!]\, n_t = 0}$$

each one can be non zero

Example

Dirichlet $\rightarrow$ $V = 0$



$\sigma = \bar{\sigma}$

Neuman

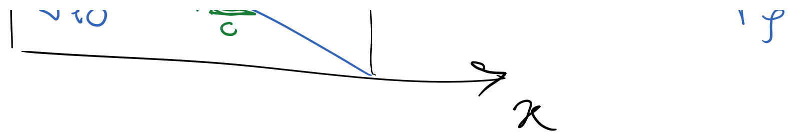
$$Z = c\rho$$

$$c = \sqrt{\frac{E}{\rho}}$$

$$n_x = \frac{1}{\sqrt{1 + c^2}}$$

$$n_t = \frac{c}{\sqrt{1 + c^2}}$$

$$n_t = \frac{c}{\sqrt{1+c^2}}$$



$$[\![\sigma]\!] = (\bar{\sigma} - 0) = \bar{\sigma}$$

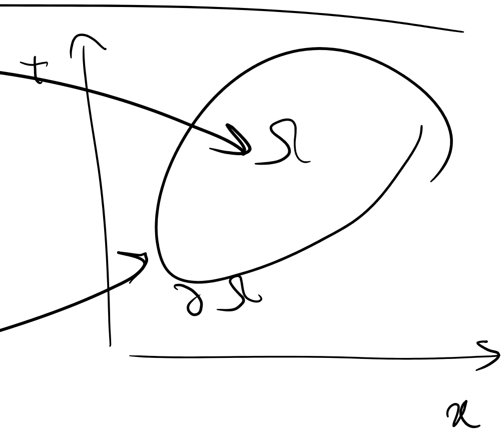
$$p = \rho v \qquad [\![p]\!] = \rho\frac{\bar{\sigma}}{Z} - 0 = \rho\frac{\bar{\sigma}}{Z}$$

$$-n_x [\![\sigma]\!] + n_t [\![p]\!] = \frac{-1}{\sqrt{1+c^2}}\bar{\sigma} + \underbrace{\frac{c}{\sqrt{1+c^2}}\rho\frac{\bar{\sigma}}{Z}}_{1} = 0$$

$$[\![\sigma]\!],\ [\![v]\!],\ [\![p]\!] \neq 0$$

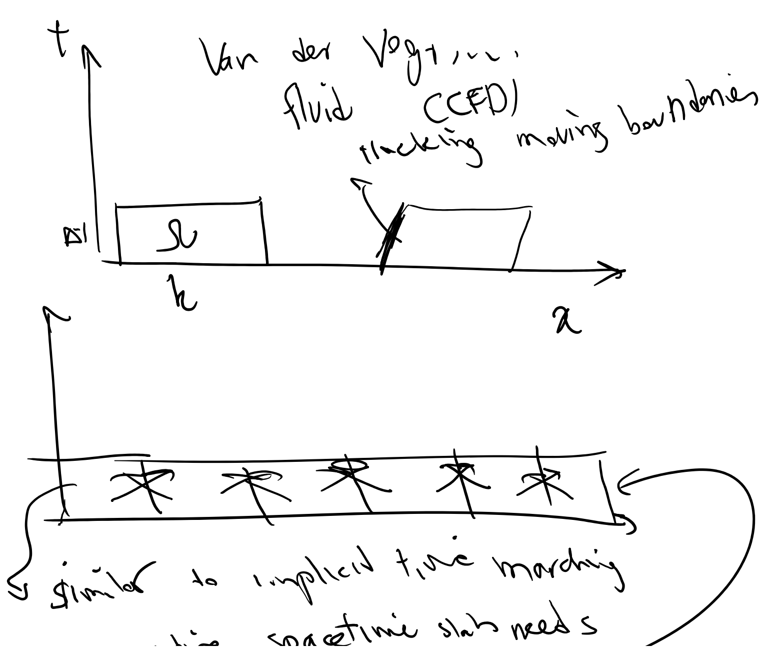
Formulating a spacetime discontinuous Galerkin method

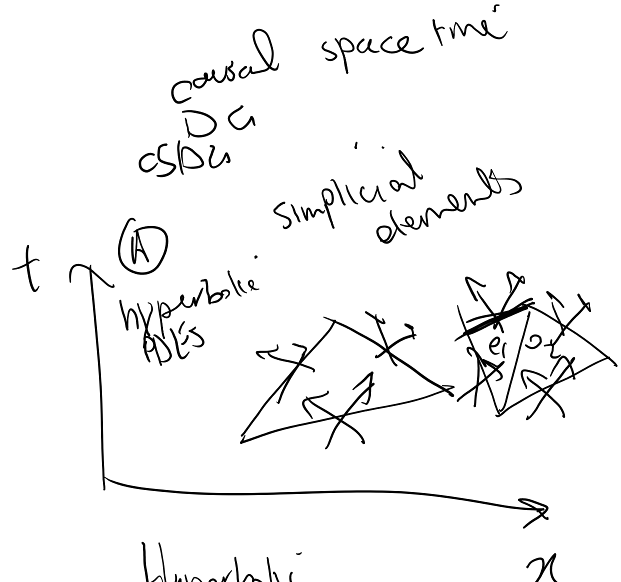
$$\begin{cases} \nabla_{st} \cdot f \sim \sigma = 0 & \text{PDE} \\ [\![F]\!] \cdot N = 0 & \text{Jump condition} \end{cases}$$



Ω's will become elements

There are two classes of elements





↳ similar to implicit time marching
as the entire spacetime slab needs to be solved at once

x

Hyperbolic

simplicial elements work for this case

two without characteristics

parabolic