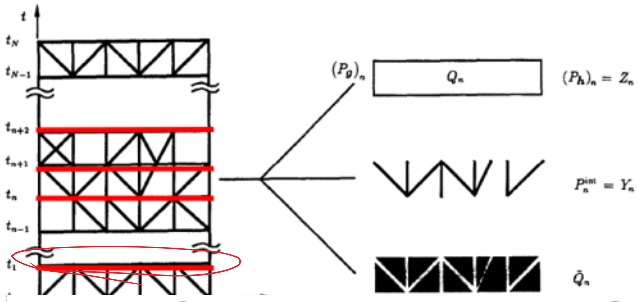
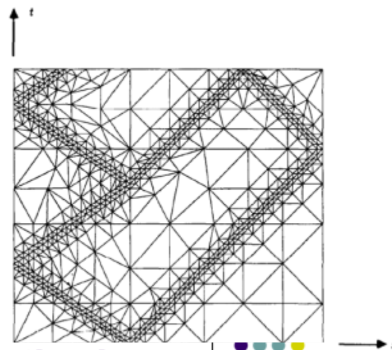
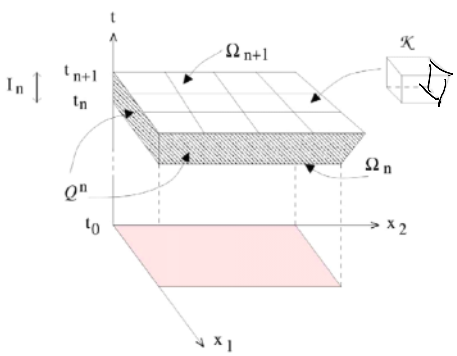
- Time Discontinuous Galerkin (TDG) methods (TJR Hughes, GM Hulbert 1987)





A few other spacetime DG methods

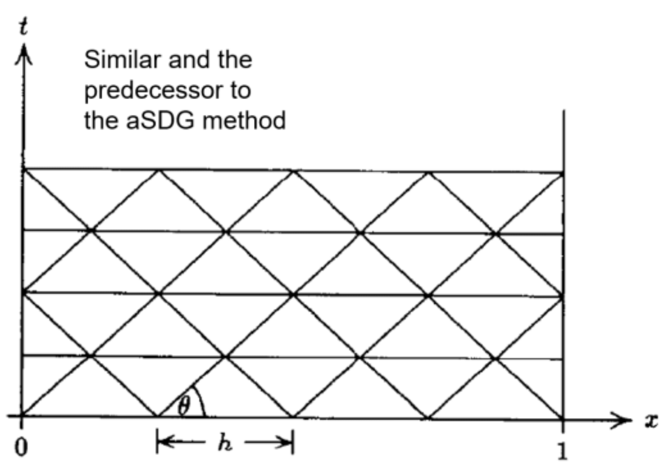
- Spacetime discontinuous Galerkin method (JJW Van der Vegt, H Van der Ven, *et al*)



- Elements are arranged in spacetime slabs
- Discontinuities are across all element boundaries
- Elements within slabs are solved simultaneously as this method is implicit

3

- Causal spacetime meshing (Richter, Falk, Lowrie, Roe, Leer, Monk, etc.)
Gerard R. Richter, An explicit finite element method for the wave equation, Applied Numerical Mathematics 16 (1994) 65-80



causal Spacetime Discontinuous Galerkin (cSDG) Finite Element Method

Discontinuous basis functions for Discontinuous Galerkin (DG) methods

DG + spacetime meshing + causal meshes for hyperbolic problems:

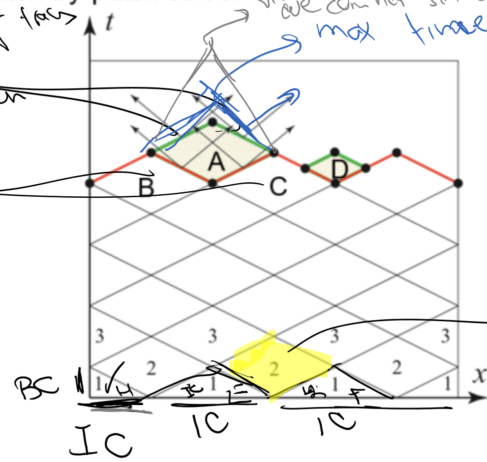
- Local solution property
- O(N) complexity (solution cost scales linearly vs. number of elements N)
- Asynchronous patch-by-patch solver

violates cacality! we can not solve element A alone

→ max time advance

* values are interior faces → element does not depend on neighbors at green faces

B, C provide * values for A



BC

IC

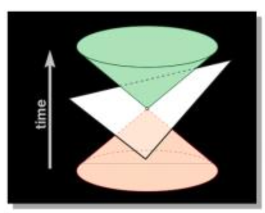
IC

IC

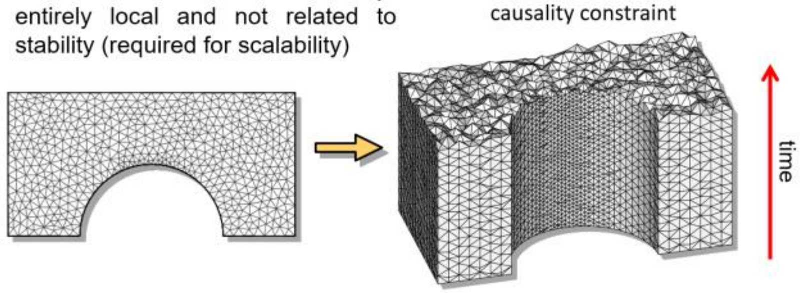
this element 2 can be solved after E, F

Tent Pitcher: Causal spacetime meshing

- Given a space mesh, Tent Pitcher constructs a spacetime mesh such that the slope of every facet on a sequence of advancing fronts is bounded by a causality constraint
- Similar to CFL condition, except entirely local and not related to stability (required for scalability)



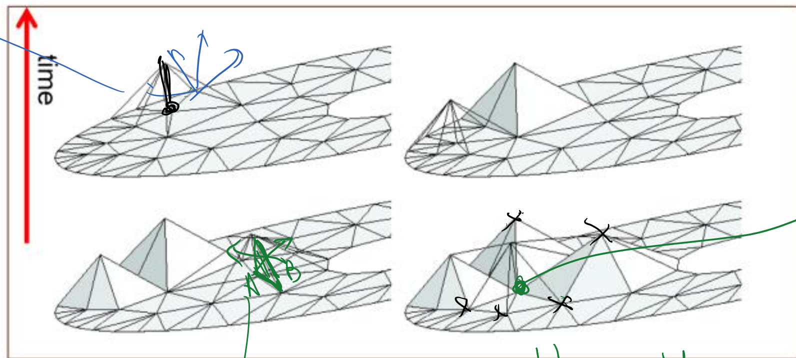
causality constraint



Tent Pitcher:
Patch–by–patch meshing

- meshing and solution are interleaved
  - patches ('tents') of tetrahedra are solve immediately ⇨ O(N) property
  - rich parallel structure: patches can be created and solved in parallel

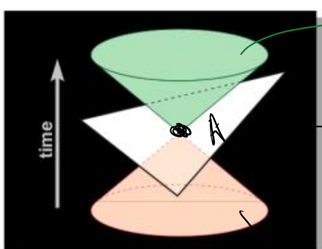
extenior facets of the patch are causal



the coordinate of the pitched vertex need to be a local minimum

elements in the patch are coupled & need to be solved simultaneously

tent–pitching sequence



causality constraint

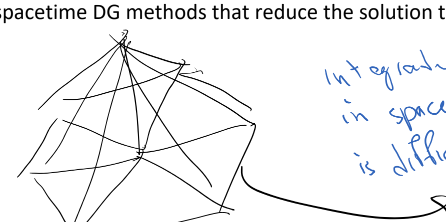
→ domain of influence
Solution at A influences the solution in green region

→ Solution of point A depends on values in domain of dependence

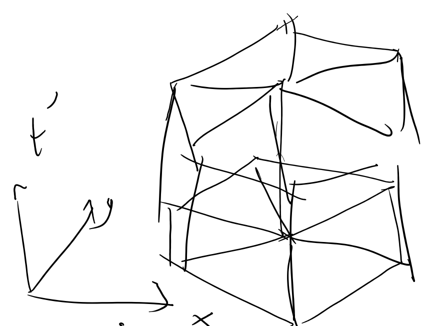
domain of dependence

There are two works on causal spacetime DG methods that reduce the solution time:

I.



Integration in space time is difficult



Prism

Parent element

* time advance is not affected by polynomial order
* At the same time conventional time marching methods are used

all conventional time stepping methods can be use inside this patch

2. Trefftz Methods

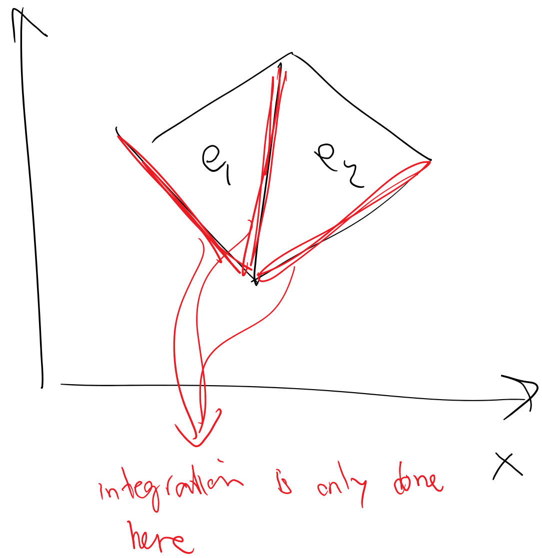
2. Treffetz Methods

$$R_i = \rho\ddot{u} + \delta\dot{u} - \nabla\cdot\sigma - \cancel{f}$$

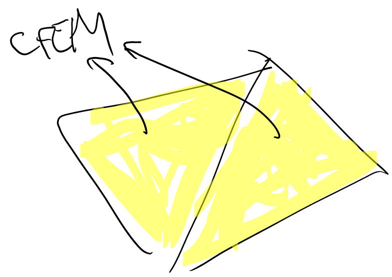
be1, e2

for zero source term

$$R_\gamma = \begin{cases} v^* - v \\ \sigma_n^* - \sigma\cdot n \end{cases}$$



With Treffetz methods the solution is interpolated with functions that already satisfy Ri -> so we don't need to weakly enforce Ri -> we only weakly enforce R on the boundary of the elements.



- Having source terms make it a bit more challenging
- Nonlinearity is a challenge for this.

Interpolation of the solution:
Recall for the time marching schemes we had:

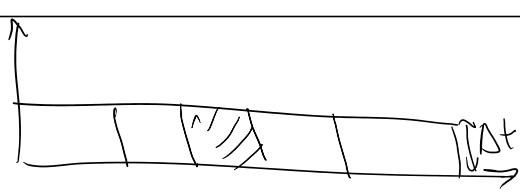
$$T(x,t) = \sum_{i=1}^{n} T_i(x)\, a_i(t) \dashrightarrow M\ddot{a} + C\dot{a} + Ka = F$$

for hyperbolic case

in spacetime

$$\boxed{T(x,t) = \sum_{i=1}^{n} T_i(x, t)\, a_i \longrightarrow Ka = F}$$

no mass matrix



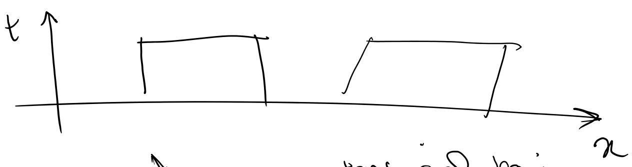
similar to implicit methods

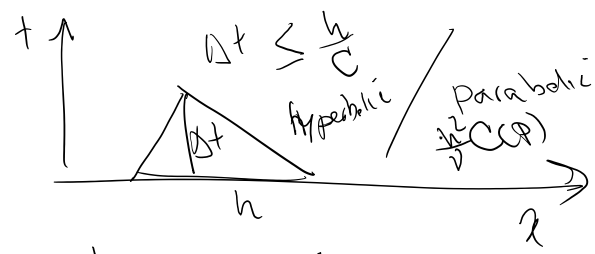
1. Δt does not have a limit for stability
2. the slab should be solved as a coupled system

Examples of basis functions

t ↑   t ↑ $\Delta t \leq \frac{h}{c}$

Examples of basis functions



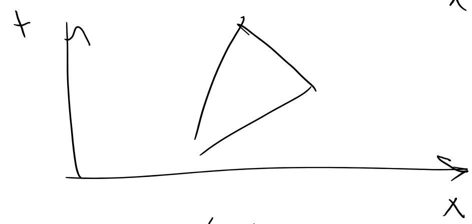


we generally use tensorial basis functions for this case:

$p = 1$ $\quad \begin{bmatrix} 1 \\ x \end{bmatrix} \otimes \begin{bmatrix} 1 \\ t \end{bmatrix}$ $\quad (1, x, t, xt)$

$p = 2$ $\quad \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}$

$P_x = 2 \quad P_t = 1 \quad \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ t \end{bmatrix} \quad \begin{pmatrix} 1 & x & x^2 \\ t & xt & x^2 t \end{pmatrix}$



$p = 0 \quad \{1\}$

$p = 1$

$$\begin{matrix} & 1 & \\ x & & t \end{matrix}$$

$p = 2$

$$\begin{matrix} & & 1 & & \\ & x & & t & \\ x^2 & & xt & & t^2 \end{matrix}$$

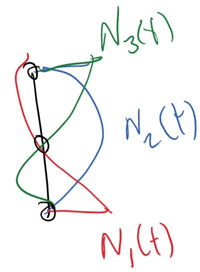
$P_x = 2 \quad P_t = 1$

$$\begin{matrix} & & 1 & & \\ & x & & y & \\ x^2 & & xy & & y^2 \end{matrix} \otimes \begin{bmatrix} 1 \\ t \end{bmatrix}$$

For these methods, we can use Finite element shape functions in space

$$\begin{matrix} & & 1 & & \\ & x & & y & \\ x^2 & & xy & & y^2 \end{matrix} \otimes$$

$P_x = 2$



$P_t = 2$

With these shape functions in time, we directly get the solution at all intermediate time values between the time steps. So, it's similar to implicit RK methods but the intermediate solutions have real meaning.

From here, working on causal spacetime methods
Heat conduction and elastodynamics

Let's start with MCV model which is a hyperbolic:

$$① \begin{cases} C\dot{T} + \nabla \cdot q = Q \\ \tau \dot{q} + k\nabla T = -q \end{cases}$$

$$① \begin{cases} CT + \nabla\cdot q = Q \\ \tau\dot{q} + K\nabla T = -q \end{cases}$$

single-log mode

$-K\nabla T$ single lage

Fourier mode $q = -K\nabla T$

$\tau\dot{q}$ delays getting to $q = -K\nabla T$ (if it ever gets there)

Conservation law form
$$\begin{cases} C\dot{T} + \nabla\cdot q = Q \\ \tau\dot{q} + \nabla\cdot KT = -q + \nabla\cdot KT \end{cases}$$

$$\tau C\ddot{T} + C\dot{T} - \nabla\cdot K\nabla T = \tau\dot{Q} + Q$$

For simplicity let $\nabla\cdot K = 0$

$c = \sqrt{\frac{|K|}{\tau C}}$ max eigenvalue of K

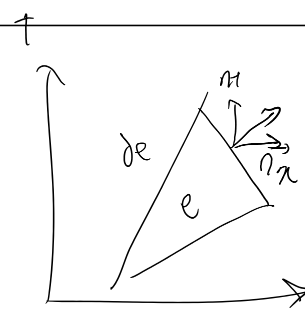
or K if $K = KI$

$$\begin{matrix} \hat{T} \longleftarrow \\ \hat{q} \longleftarrow \end{matrix} \begin{cases} C\dot{T} + \nabla\cdot q - Q = 0 \\ K^{-1}\tau\dot{q} + \nabla T = -K^{-1}q \end{cases}$$

WRS

$$\int_e \left[\underbrace{\hat{T}(C\dot{T}}_{①} + \underbrace{\nabla\cdot q}_{②} - Q) + \underbrace{\hat{q}(K^{-1}\tau\dot{q}}_{③} + \underbrace{\nabla T}_{④} + K^{-1}q)\right] dV$$

PDEs from the balance law



$$\int_{\partial e} \left\{\hat{T}(C\overset{*}{T} - CT)n_t + \hat{T}(\overset{*}{q}_n - q\cdot n)n_x + \left(\hat{q}\,K^{-1}\tau(\overset{*}{q} - q)n_t + \hat{q}(\overset{*}{T} - T)n_x\right)\right\} ds$$

① ② ③ ④

Jump conditions from the balance law

to get the weak statement $\Longrightarrow$ Gauss theorem

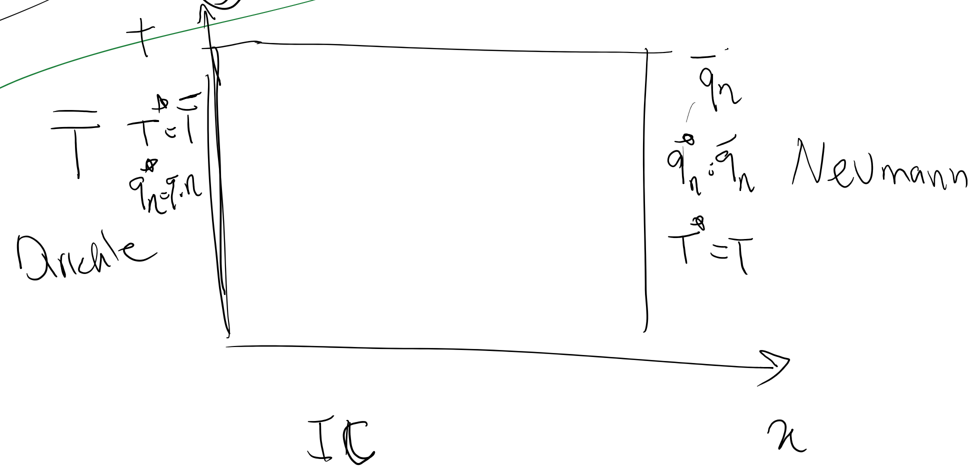
$$\int_e \left(-\dot{\hat{T}}CT - \nabla\hat{T}q - \hat{T}Q - \dot{\hat{q}}K^{-1}\tau q \quad \nabla\hat{q}\,T + \hat{q}K^{-1}q\right) dV$$

$$+\int_{\partial e} \left(\underbrace{\hat{T} C T^{*} n_t}_{①} + \underbrace{\hat{T} q_n^{*} n_x}_{②} + \underbrace{\hat{q} k^{-1} \tau q^{*} n_t}_{③} + \underbrace{\hat{q} T^{*} n_x}_{④} \right) dS = 0$$

WK

IC

BC



$q^{*} = q_{IC}$ $T^{*} = T_{IC}$