2022/11/15

Tuesday, November 15, 2022 9:44 AM

From last time

$$\int_e \left[\hat{T}\left(c\dot{T} + \nabla_x q - Q\right) + \hat{q}\left(k^{-1}\tau\dot{q} + \nabla T + k^{-1} q\right)\right] dV$$

$$\int_{\partial e} \hat{T}(c T^* - cT) n_t + \hat{T}(q_n^* - q\cdot n) n_x + \left(\hat{q}\, k^{-1}\tau (q^* - q) n_t + \hat{q}(T^* - T) n_x \; ds$$

weak statement (WK-H)

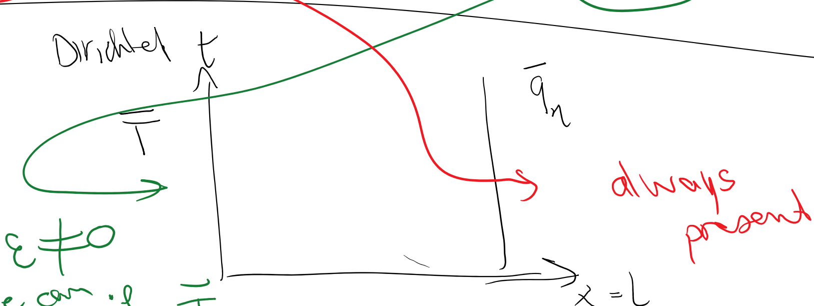
$$\int_e \left(-\dot{\hat{T}} c T - \nabla\hat{T} q - \hat{T} Q\right) + \varepsilon \hat{q}\underbrace{\left(k^{-1}\tau\dot{q} + \nabla T + k^{-1} q\right)}_{\text{const eqn}} dV$$

$$\int_{\partial e}\left\{\hat{T} c T^* n_t + \hat{T} q_n^* n_x + \varepsilon\hat{q}\, k^{-1}\tau(q^*-q)n_t + \varepsilon\hat{q}(T^*-T)n_x\right\} ds = 0$$

parabolic $\tau = 0$ — 2F like above; 1F $q = -k\nabla T$, constitutive eqn goes away

parabolic 1F

$$\int_e\left(-\dot{\hat T} c T - \nabla\hat T q - \hat T Q\right) dV + \int_{\partial e}\left\{\hat T c T^* n_t + \hat T q_n^* n_x + \varepsilon \hat q\, k^{-1}\tau(q^*-q)n_t + \varepsilon\hat q(T^*-T)n_x\right\} ds = 0$$



For 1F parabolic case:

Natural Boundary condition (red term) can always be enforced

Essential BC:

- We need $\varepsilon \neq 0$
- Or for $\varepsilon = 0$ Refer to fluxes for parabolic problem

$$\hat{q} = \{q\} + \frac{\eta_0 k}{h}[T]$$

$$\hat{T} = \{T\}$$

or alternating q^+, T^-

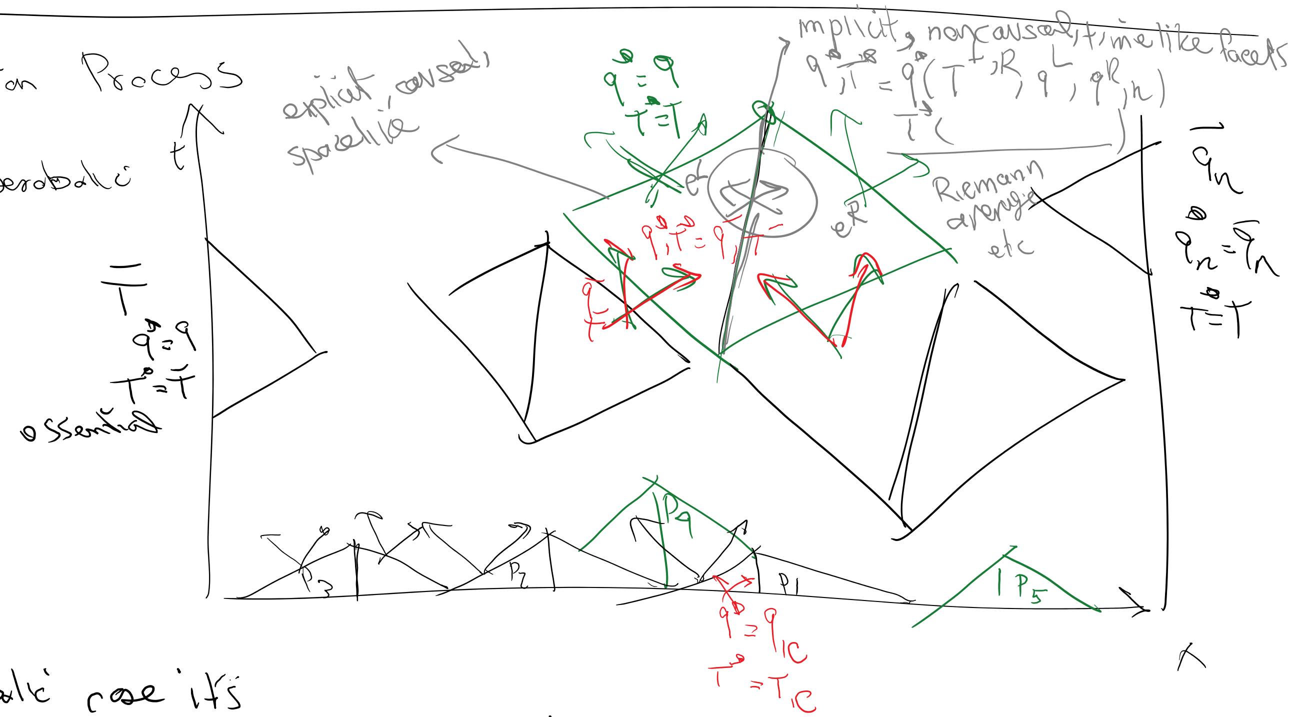
we can specify $\bar{T}$ $\longrightarrow$ $x = L$

for $\varepsilon = 0$ we need $\alpha \neq 0$ so essential BC can be specified

- Unlike elliptic case, the alpha term is not needed for stability. We just need this for enforcing essential BC when $\varepsilon = 0$

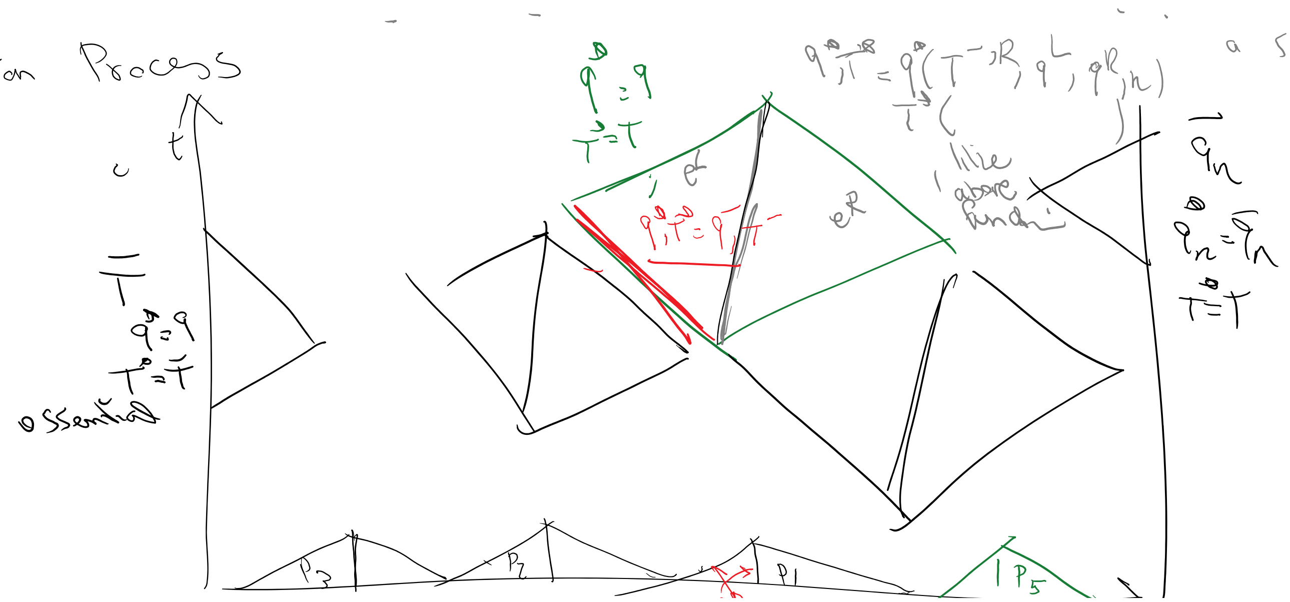
Solution Process

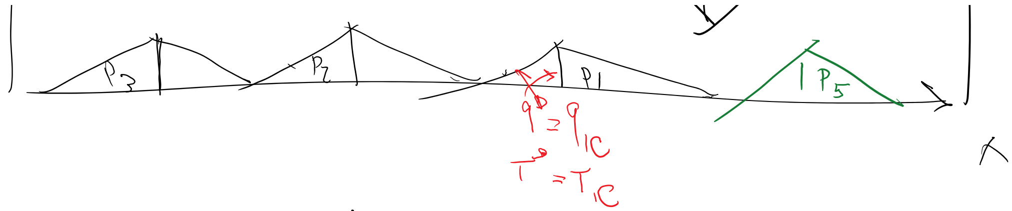
Hyperbolic



Parabolic case it's similar but there is no characteristics

Solution Process





- For any nonhorizontal facet, the star value really depends on the two sides values.
- For inflow (red) we can use this because the inflow solution is available and the time outflow side is being solved. -> **We make the approximation to let the star equal to the earlier solution**.
- For **outflow** (green) we cannot have a local solution process if the star value is similar to interior facets (i.e. depending on the two sides). We **make an approximation by letting the star equal to the interior trace**.

The parabolic case is stable if the time advance is limited by element size^2.

Similar background:

An explicit discontinuous Galerkin scheme with local time-stepping for general unsteady diffusion equations

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Ghost element: temporary sol'n is obtained by extrapolation in time

first element solved

it has two ghost element neighbors

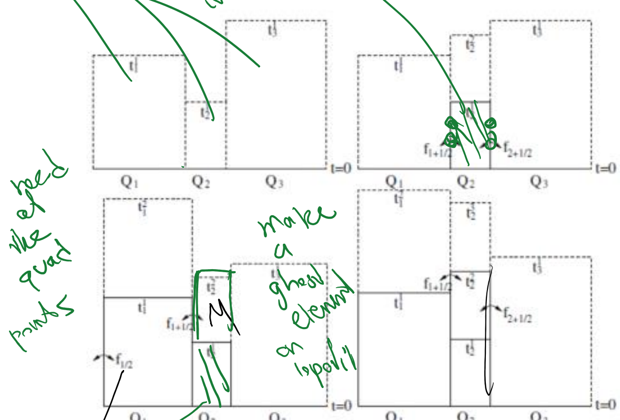
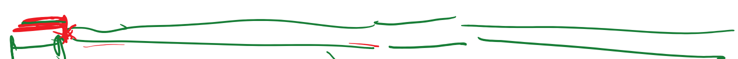


Fig. 1. Sequence of steps 1–4 of a computation with three different elements and local time-stepping.

permanent



Smallest element limits the global time advance

this element has BC, real element & ghost neighbor elements along its vertical boundaries & can be solved

ADER DG is a very powerful method: Like cSDG it's

- Arbitrarily high order in space and time
- Asynchronous

But here are some concerns:

- For nonlinear PDEs (like fluids the extrapolation in time is very complicated)
- For both hyperbolic and parabolic PDEs the time step is drastically influenced by polynomial order. In SDG, the time advance is not influenced by polynomial order in hyperbolic case.

correction factor

$$\Delta t \leqslant \beta(N)\frac{\Delta x^2}{(2N+1)^2\mu\sqrt{d}}, \qquad (3.42)$$

element size

d : space dimension

polynomial order

space and ~~time~~ time

ignore β

$\dot{T} - \mu \Delta T$

$\frac{L^2}{T}$

error $\propto (\Delta t)^{N+1}$ or $(\Delta t)^N$

$N = 1 \quad \frac{1}{1^2}\beta$

$N = 5 \quad \frac{1}{11^2}\beta \quad \sim 100x$ smaller time step

Table 2
Stability numbers of the optimized STE-DG scheme

N	1	2	3	4	5	6	7
β_{max}	1.46	0.8	0.54	0.355	0.28	0.21	0.16
β_{min}	1.0	0.33	0.20	0.14	0.10	0.08	0.06
$\tilde{\beta}_{max}$	2.8	0.86	0.40	0.24	0.16	0.12	0.09

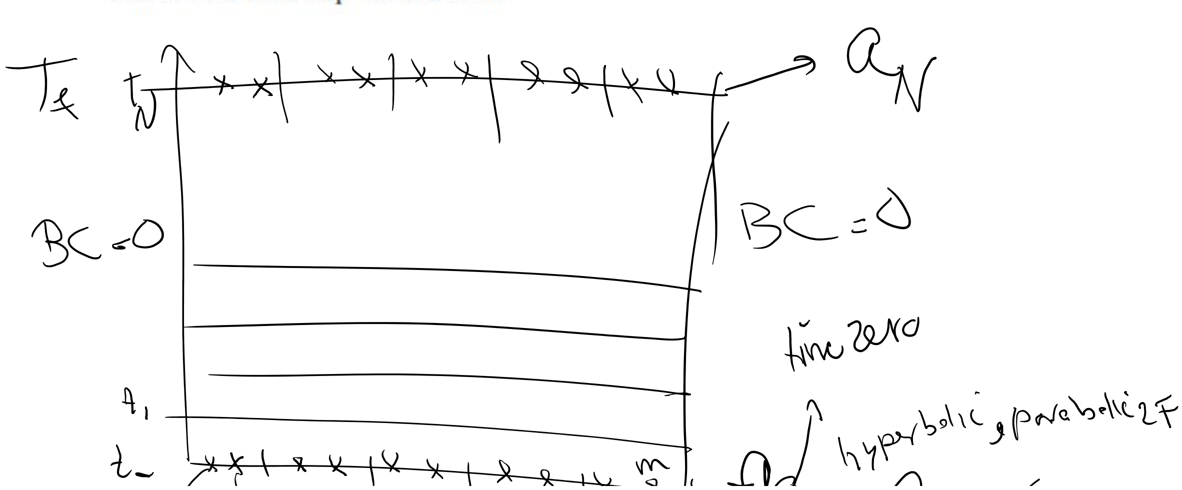
Global stability analysis of linear PDEs

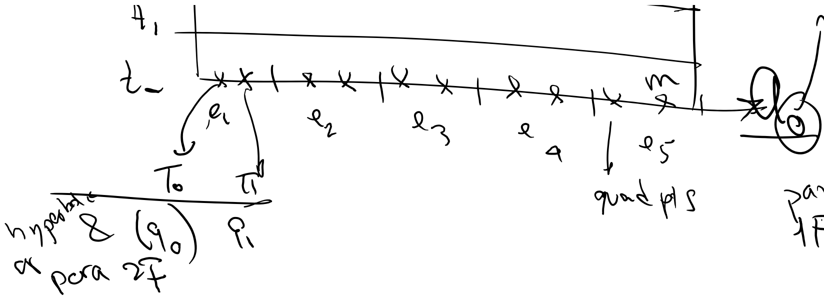
To analyze the stability of the scheme, a periodic problem with a given spatial discretization is considered. As the problem (3.40) is linear, one can construct a matrix $\boldsymbol{W}$ such that

$$\hat{u}^{new} = \boldsymbol{W}\hat{u}^{old}, \qquad (3.41)$$

where $\hat{u}^{old}$ denote DOF at a common time level t^{old} and $\hat{u}^{new}$ denote DOF at a common time level t^{new} with $t^{new} > t^{old}$. $\boldsymbol{W} = \boldsymbol{W}(\Delta t_i)$ depends on the time steps Δt_i of each element Q_i. Following the matrix method of stability analysis described in [14], the scheme is stable, if the spectral radius $\rho(\boldsymbol{W})$ is lower or equal to 1.

We first consider uniform grid spacing and uniform polynomial order, thus, $\Delta x = \text{const}$ and $\Delta t = \text{const}$ in the whole computational domain. For an explicit DG scheme discretizing equation (3.40), a stability restriction for the time step has the form





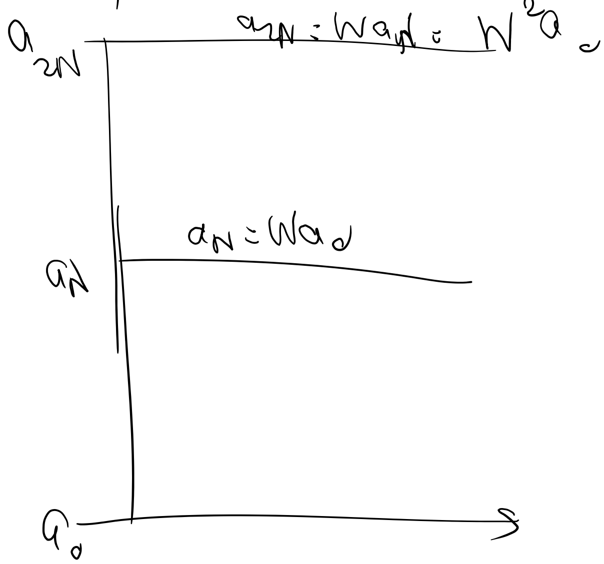
hyperbolic, parabolic 2F $a = (T_0, q_0, T_1, q_1, \ldots, T_m, q_m)$

parabolic 1F $a \quad (T_0, T_1 \quad T_m)$

linear PDE size M

$$a_{N} = f(a_0)$$

$$\boxed{a_N = W a_0}$$



$$\boxed{a_{qN} = W^q a_0}$$

Eigen decomposition (if it exists) $WU = U\Lambda$

$$U = [u_1 | u_2 | \ldots | u_M] \qquad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$$

right eigenvectors

$$\left.\begin{aligned} W &= U \Lambda U^{-1} \\ a_{qN} &= W^q a_0 \end{aligned}\right\} \rightarrow \boxed{a_{qN} = U \Lambda^q U^{-1} a_0}$$

for all λ_i : $|\lambda_i| < 1$

$\|a \ldots\| < \|x \ldots\|$

for all Λ_i : $|\lambda_i| \leq 1$ $\quad \|a_{qN}\| \leq \|a_o\|$

In general W may not be diagonalizable

$\longrightarrow$ We can always reduce it to Jordan form

$$\underset{M \times M}{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & 1 & \\ & & \lambda_2 & \\ & & & \ddots \end{bmatrix}$$

$\rho(W)$

$\hookrightarrow$ spectral radius $= \underset{i=1 \dots M}{\text{Min}} (|\lambda_i|)$

For stability

$$\boxed{\begin{array}{ll} \rho(W) < 1 & \text{in general} \\ \rho(W) \leq 1 & W \text{ is diagonalizable} \end{array}}$$

$\rho(W) = 1$ & nondiagonalizable $\longrightarrow$ weak unstable

linearly blows up

$\rho(W) > 1 \longrightarrow$ strong unstable / exponentially blows up