2022/11/29

Tuesday, November 29, 2022 10:07 AM

Continuing the stability discussion

1. Approach 1 (Global spectral stability) - Discussed before - Linear PDE
2. Approach 2 (Local spectral stability) - Linear
3. Von Neumann analysis - Linear
4. Energy balance -> can be applied to nonlinear ones too

2. Local spectral stability

$$CT_{,t} + q_{,x} = Q \tag{1a}$$

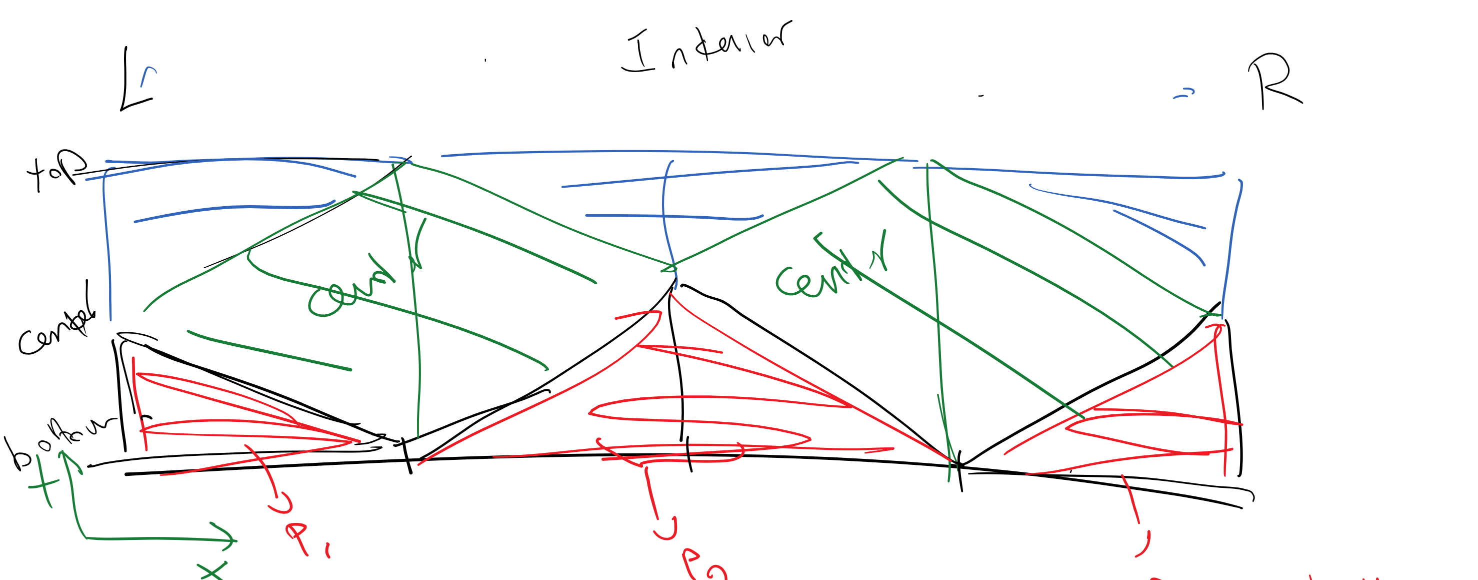
$$\tau q_{,t} + \kappa T_{,x} = -\eta q, \tag{1b}$$

Table 1: The classification of PDEs corresponding to (1).

PDE and symbol	conditions	Example
Parabolic (P)	$\tau = 0,\ \eta = 1$	Fourier heat conduction
Damped hyperbolic (dH)	$\tau > 0,\ \eta = 1$	MCV
Undamped hyperbolic (H)	$\tau > 0,\ \eta = 0$	Wave equation

Weak statement:

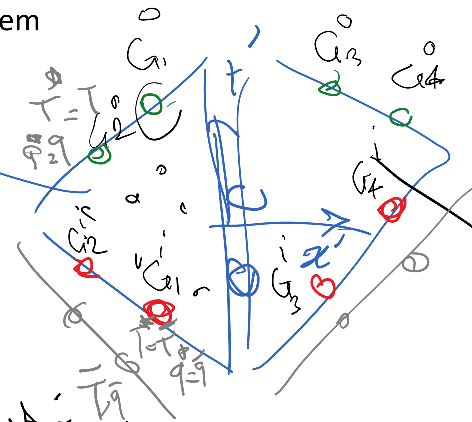
$$\int_e \Big(-\hat{T}_{,t} CT - \hat{T}_{,x} q - \hat{T} Q - \kappa^{-1}\tau \hat{q}_{,t} q - \hat{q}_{,x} T + \kappa^{-1}\eta \hat{q} q \Big) \mathrm{d}V +$$
$$\int_{\partial e} \Big(\hat{T}\Big[CT^* n_t + q_n^* \Big] + \hat{q}\Big[\kappa^{-1}\tau q^* n_t + T^* n_x \Big] \Big) \mathrm{d}S = 0$$



9 cases

If we can solve these 9 shapes, we can solve any 1D problem

T 2nd order
1st element



12 solution coefficients

$$T = [1 \;\; x' \;\; t' \;\; x'^2 \;\; x't' \;\; t'^2] \begin{bmatrix} T_1^s \\ T_2^s \\ T_3^s \\ \vdots \\ T_6^s \end{bmatrix}$$

q = "

$$\underbrace{\begin{bmatrix} q_1^s \\ \vdots \\ q_6^s \end{bmatrix}}_{\text{solution coefficients}}$$

$P_T = 2$

24 solution coefficients for the patch

O = order of integrand $= 2P_T$

Number of Gauss pts = $\mathrm{ceil}\left(\frac{O+1}{2}\right) = P_T$

Input Inflow: 8 $\quad a_i \quad [\bar{T}(G_1^i), \bar{q}(G_1^i) \; \ldots \;, \; \bar{q}(G_4^i)]$

24 Solution coefficients $\quad [\overbrace{T_1^s - - T_6^s}^{e_1} \; \overbrace{q_1^s \to q_6^s}^{e_1} \; \overbrace{T_1^s \to T_6^s}^{e_2} \; \overbrace{q_1^s \to q_6^s}^{e_2}]$

a_s

Outflow 8 $\quad [T(G_1^o) \; q(G_1^o) \; \ldots \; \ldots \; q(G_4^o)]$

a_o

(1) $\underset{24\times1}{a_s} = \underset{24\times8}{T_{i\to s}} \; \underset{8\times1}{a_i}$

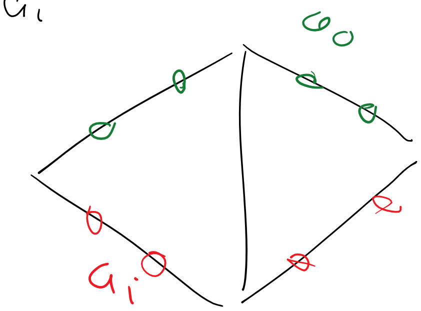
Can get $T_{i\to s}$ each time one of a_i's = 1 others zero can get $T_{i\to s}$ each time one of a_i's $=1$ others zero, j'th

solve as column j of $T_{i\to s}$

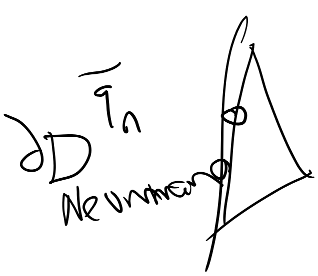
② $\underbrace{a_o}_{8\times 1} = \underbrace{T_{s\to o}}_{8\times 24}\ \underbrace{a_s}_{24\times 1}$

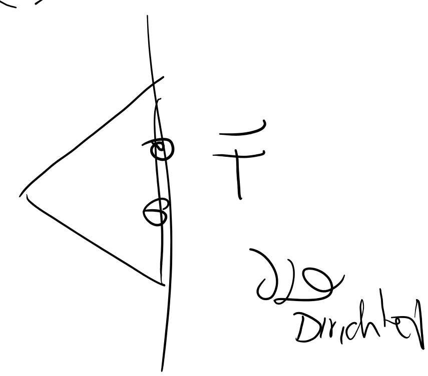
$a_o = T_{s\to o}\, T_{i\to s}\, a_i$

$$\boxed{a_o = T_{i\to o}\, a_i}$$



Do this for 8 more shapes



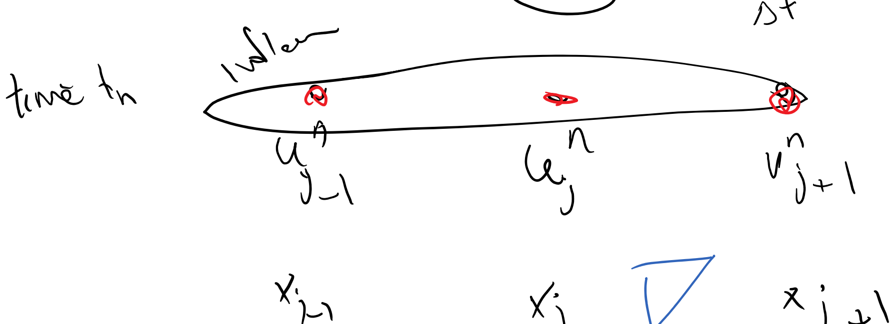


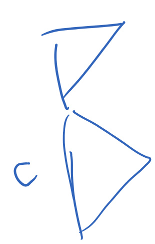
This is like finite difference (FD)
But you can make it as high order needed in space and time

$\dot{u} - u_{,xx} = 0$

unknown u_j^{n+1}

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n + u_{j-1}^n - 2u_j^n}{\Delta x^2}$$





You can solve different problems this way

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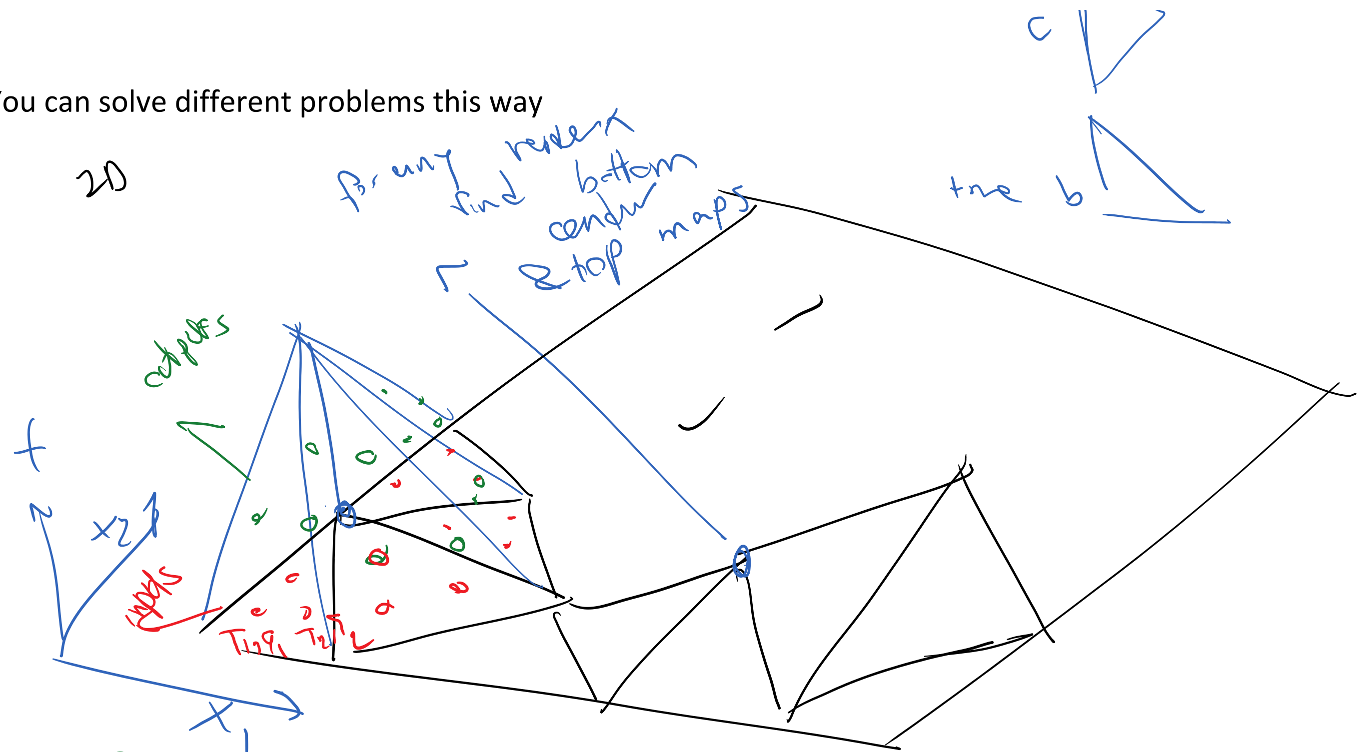
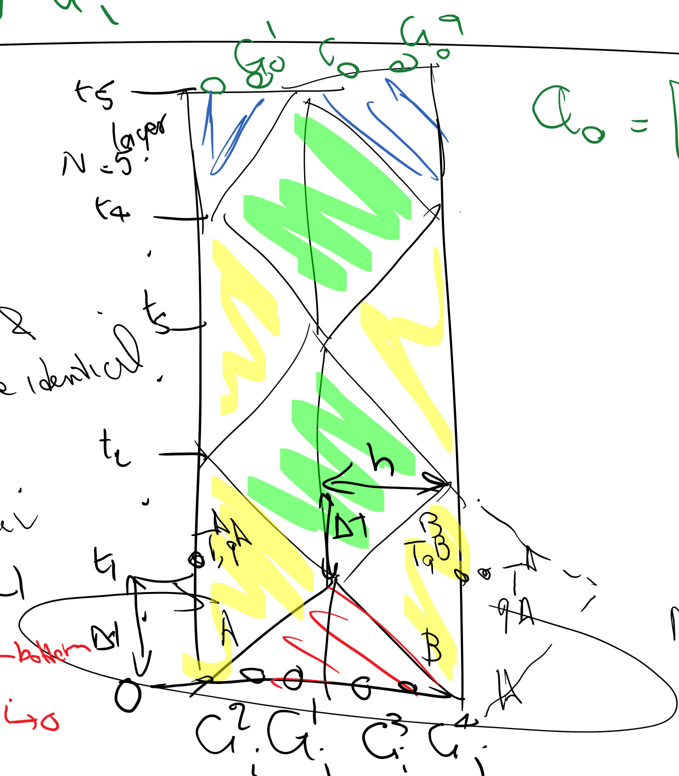
2D



$$a_0 = T a_i$$

Stability



$$a_0 = [T(G_0^1), \ldots, q(G_0^4)]$$

map of green & yellow are identical

periodic

PBC
Numerical flux is for
$(T,q)^\ell = (T,q)^B$
$(T,q)^r = (T,q)^A$

$$T_{i\to 0} = T^{top}_{i\to 0}(T^{center}_{i\to 0})\, T^{bottom}_{i\to 0}$$

$$a_i = [T(G_i^1), q(G_i^1), \ldots\ \ldots\ q(G_i^4)]$$

$\rho(T_{i\to 0}) < 1 \longrightarrow$ how large Δt can be for a given h

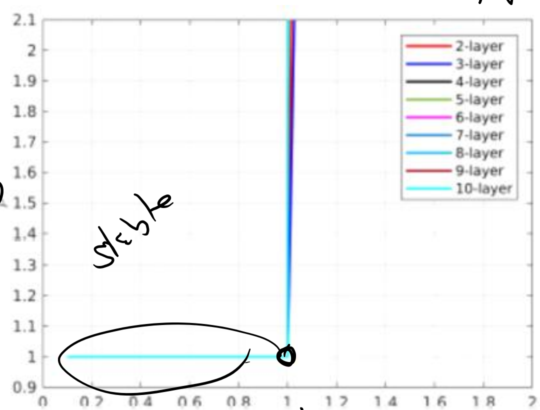
Hyperbolic:

$$\Delta t = \alpha \frac{h}{c}, \quad c \rightarrow \text{wave speed}$$

$\alpha = 1$ (CFL = 1)

independent of polynomial order

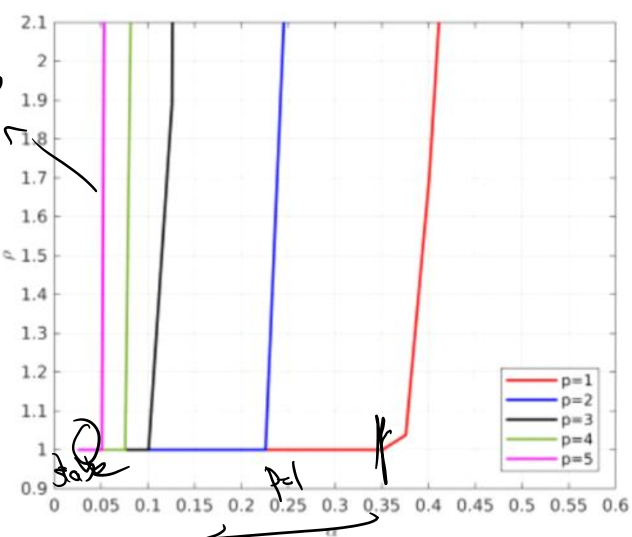
N



Parabolic

$$C\dot{T} - k T_{,xx} = Q$$

$$\Delta t = \alpha(\rho_T)\frac{h^2}{D} \qquad \dot{T} - D T_{,xx} = \qquad D = \frac{k}{C} \quad \left[\frac{L^2}{T}\right]$$



Parabolic equation

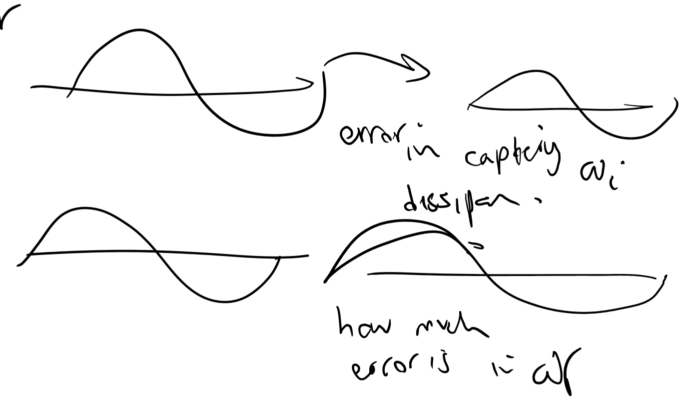
p	1	2	3	4	5
$\alpha(p)$	0.35	0.225	0.1	0.075	0.05

Similar to approach 1

3. Von Neumann Very powerful for linear PDEs

Dispersion — Dissipation error

— Dispersion error



— Stability limit

Idea

Wave equation
$$\begin{cases} C\dot{T} - q_{,x} = Q \\ \tau\dot{q} - \kappa T_{,x} = 0 \end{cases}$$

$$\boxed{\tau C\ddot{T} - \kappa T_{,xx} = 0}$$

Dispersion analysis

Source term $Q = 0$

exact solution

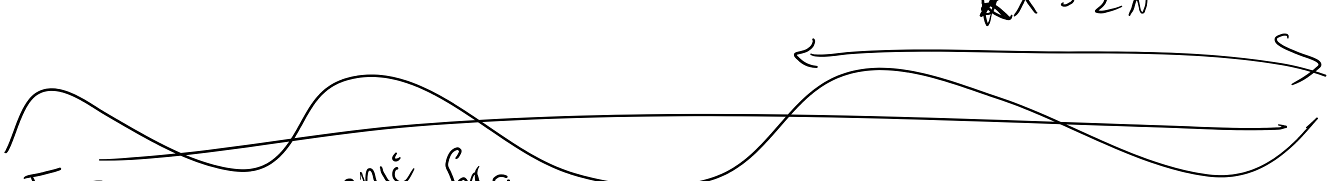
$$T = e^{i(Kx - \omega t)}$$

seek harmonic solutions

given

sought

$K\lambda = 2\pi \quad K = \frac{2\pi}{\lambda}$



IC harmonic for a given k

IC harmonic for a given k