2022/12/06

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Dispersion error analysis

Von Neumann limits
- Linear PDE
- Homogeneous material

3 cases

(1) parabolic

$$\begin{cases} C\dot{T} + \nabla \cdot q = Q \\ -K\nabla T = q \end{cases}$$

$$C\dot{T} - \nabla \cdot K\nabla T = Q$$

dispersion analysis $Q = 0$

1D, also K is constant

$$C\dot{T} - K T_{,xx} = 0 \qquad T = e^{i(kx - \omega t)}$$

k given $\to$ find ω

$$\left(-C(i\omega) + K k^2\right) e^{i(kx - \omega t)} = 0$$

$$\boxed{\omega = -i\frac{K k^2}{C}}$$ 1a Parabolic case

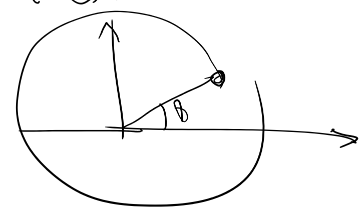
wave propagation part

θ

$$T = e^{i(kx - \omega t)} = e^{i(kx - \omega_R t)} e^{i(-i\omega_I t)} = \underbrace{e^{i(kx - \omega_R t)}}_{\theta} e^{\omega_I t}$$

$$|T| = e^{\omega_I t}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$



For the solution not to blow up

$$\omega_I t \leq 0 \qquad \boxed{\omega_I \leq 0}$$

$= 0$ conservative case

The actual solution stability is called Dynamic Stability

Wave equation

1D

$$\begin{cases} C\dot{T} + q_{,x} = Q \\ \tau\dot{q} - K T_{,x} = \cancel{X}\ 0 \end{cases}$$

$$C\tau\ddot{T} - K T_{,xx} = 0 \qquad \boxed{\ddot{T} - c^2 T_{,xx} = 0} \qquad c = \sqrt{\frac{K}{C\tau}}$$

$$T = e^{i(kx - \omega t)}$$

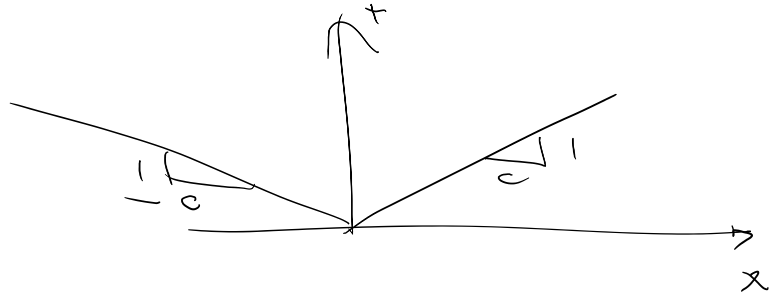
$$\left(\quad c^2 k^2 \right) e^{i(kx - \omega t)}$$

$$T = e^{i(kx-\omega t)} \longrightarrow \left(-\omega^2 + c^2 k^2\right) e^{i(kx-\omega t)} = 0$$

$$\boxed{\omega = \pm ck} \quad (1b)$$

all real no Dissipation

this corresponds to waves going to left & right with speed c



MCV

$$\begin{cases} C\dot{T} + q_{,x} = \cancel{Q}^{\,0} \\ \tau\dot{q} - \kappa T_{,x} = q \end{cases} \longrightarrow C\tau\ddot{T} + C\dot{T} - \kappa T_{,xx} = 0$$

$$\ddot{T} + \frac{1}{\tau}\dot{T} - c^2 T_{,xx} = 0$$

new term

$\ell = c\tau$ characteristic length scale

$$\boxed{\ddot{T} + \frac{c}{\ell}\dot{T} - c^2 T_{,xx} = 0}$$

$$T = e^{i(kx-\omega t)}$$

$$-\omega^2 - i\frac{c}{\ell}\omega + c^2k^2 = 0 \quad \times \tau^2$$

$$(\omega\tau)^2 + i\frac{c\tau}{\ell}(\omega\tau) - (\tau c)^2 k^2 = 0$$

$$\boxed{(\omega\tau)^2 + i(\omega\tau) - (\ell k)^2 = 0}$$

$$\boxed{\omega\tau = \frac{-i \pm \sqrt{\underbrace{-1 + 4(\ell k)^2}_{\Delta}}}{2}} \quad 1c$$

Dispersion for MCV

critical k: k_c : $-1 + 4(\ell k_c)^2 = 0$

$k \le 0 \le k_c$

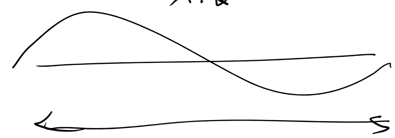
$\Delta < 0$ fully imaginary solution

$k > k_c$ complex solution $\longrightarrow$ damped propagation wave

Limits of (1c)

$\ell k \ll 1 \qquad k \ll \frac{1}{\ell} \qquad \frac{2\pi \ell}{\lambda} \ll 1$

$\lambda k = 2\pi$



$$\boxed{\lambda \gg 2\pi\ell}$$

λ

$$\omega\tau = \frac{-i \pm i\sqrt{1-4(k\ell)^2}}{2} = i\frac{(-1 \pm (1 - \frac{1}{2}\cdot 4(k\ell)^2 \cdots))}{2}$$

$(1+x)^{\frac{1}{2}} \approx 1+\frac{x}{2}$

$x \ll 1$

$(+)\quad i\frac{(-1+1-2(k\ell)^2)}{2}$

$(-)\quad i\frac{(-1-1+2(k\ell)^2)}{2}$

$$\omega\tau = \begin{cases} -i(k\ell)^2 \\ i(-1+(k\ell)^2) \end{cases}$$

for high wavelengths we're back to parabolic dispersion

$$\omega = -\frac{\kappa k^2}{C}$$

Other limit $k\ell \gg 1$

$$\omega\tau = \frac{-i \pm \sqrt{-1+4(\ell k)^2}}{2} \approx \frac{-i \pm 2k\ell}{2} \approx \pm k\ell$$

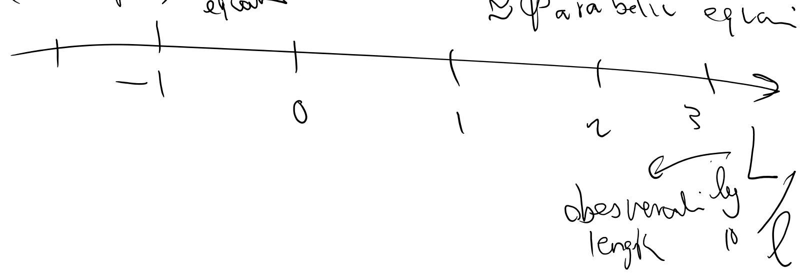
$$\omega = \pm k\left(\frac{\ell}{\tau}\right) = \pm kc$$ Undamped wave equation

damped wave equation

$$\begin{cases} C\dot{T} + \nabla\cdot q = Q \\ \tau\dot{q} - \kappa\nabla T = q \end{cases}$$

(undamped) wave equation

$\approx$ Parabolic equation



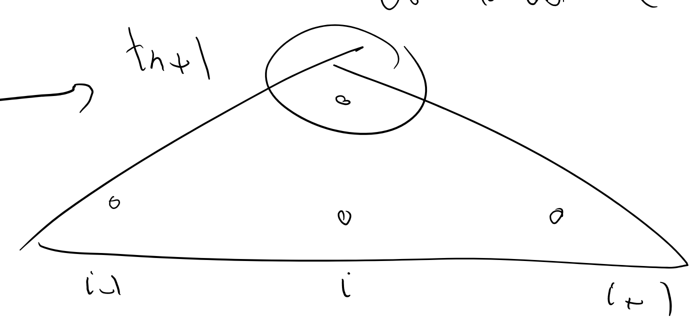
Next step:
We do the dispersion analysis for the numerical method

numerical

FD $\quad e^{i(kx - \omega^h t)}$

plug in FD update equation

ω^h is obtained



Numerical relation: $e^{i(kx - \omega_r^h t)} \; e^{\omega_I^h t}$ → obtained for FD, ...

Real PDE: $e^{i(kx - \omega_r t)} \; e^{\omega_I t}$

— Dispersion error:

$$* \quad \Delta\omega = \omega^h - \omega = (\omega_r^h - \omega_r) + i(\omega_I^h - \omega_I)$$

$\omega_r^h - \omega_r$ Dispersion error (again)

: angular frequency / wave speed, is affected

$$c = \frac{\omega}{k} \qquad \Delta c = \frac{\Delta\omega_r + \Delta\omega_I \, i}{k} = \frac{\Delta\omega_r}{c} + i\frac{\Delta\omega_I}{c}$$

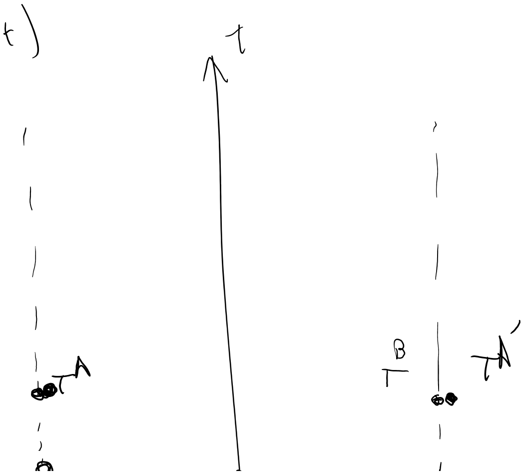
— $\omega_I^h - \omega_I$ Dissipation error

— Stability of Numerical method (Linear PDE)

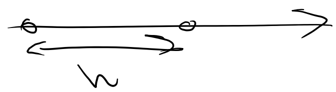
$$\boxed{\omega_I^h \leq 0}$$

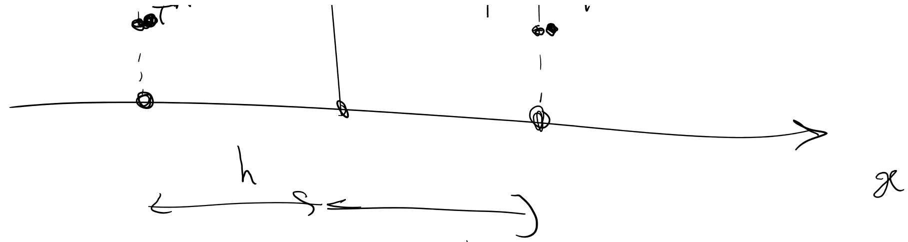
How do we do this analysis for DG and FEMs?

$$T^h = e^{i(kx - \omega^h t)}$$



$\omega^h = ?$

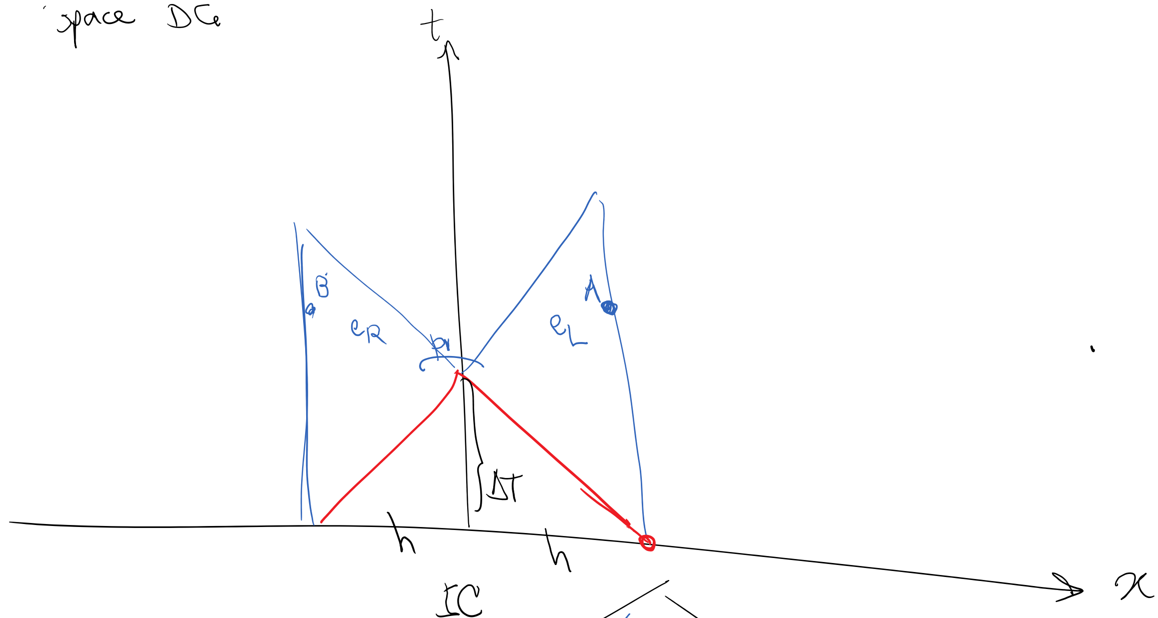


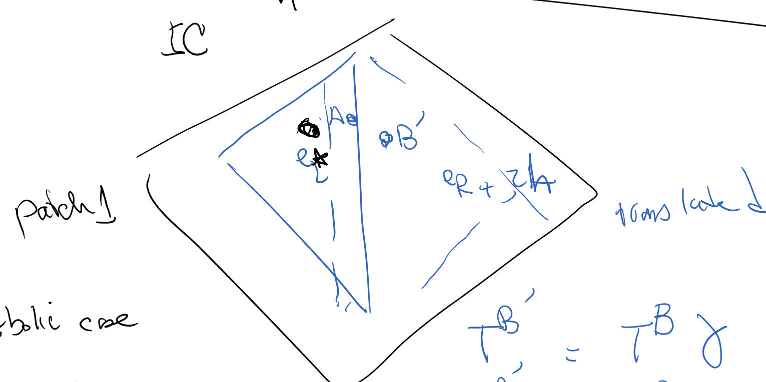


$$T^{A'} = e^{iK(2h)}\, T^{A}$$

IC $\qquad e^{ikx}$

For space DG





solve P1 $\qquad$ hyperbolic case

$$T^{B'} = T^{B}\gamma \qquad \gamma = e^{iK(2h)}$$
$$q^{B'} = q^{B}\gamma$$

$$T^{*} = \frac{T^{A} + T^{B'}}{2} + \frac{q^{A} - q^{B'}}{2Z}$$

$$q^{*} = \frac{q^{A} + q^{B'}}{2} + \frac{Z}{2}\left(T^{A} - T^{B'}\right)$$

$$\left\{\begin{array}{l} T^{*A} = \dfrac{T^{A} + \gamma T^{B}}{2} + \dfrac{q^{A} - \gamma q^{B}}{2Z} \\ \text{same for } q^{*A} = \dfrac{q^{A} + \gamma q^{B}}{2} + \dfrac{Z}{2}\left(T^{A} - \gamma T^{B}\right) \end{array}\right.$$

$$T^{*B} = \frac{1}{\gamma}\, T^{*A}$$

$$T^{*\beta} = \frac{1}{\gamma} T^{*A}$$
$$q^{*B} = \frac{1}{\gamma} q^{*B}$$

k affects M, C, K

Dispersion error analysis is done two ways:

1. Semi-discrete: Space is discretized -> $e^{i(kx-\omega t)}$ $\quad M\ddot{a} + C\dot{a} + Ka = \cancel{F}$

   $\Delta\omega(h, p_x)$ $\quad a = e^{-i\omega t}$ $\quad$ find ω

   No time marching $\quad$ Dispersion error is only from spatial discretization

2. Fully discrete:
   Do the actual time advance: Implicit vs explicit, time step size, …

   $$\Delta\omega(h, p_x, \Delta T, RK4, \ldots)$$

For DG & FEM often the analysis is semi-discrete

- DG there are a few full dispersion analysis (Space DG + time marching e.g. RK4)

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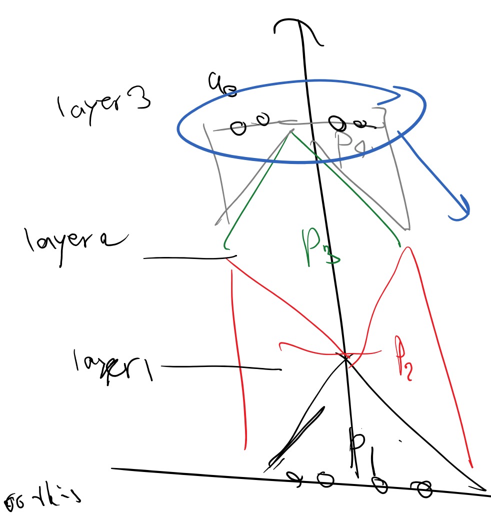
for SDG

we can use

- the maps from last time to get a_0

  $a_0 = \sum_{i \to 0} a_i$ → formula for this

- or solve the system by brute force (need a complex valued DG solver)



or even more layers

Do a best fit to $e^{i(kx - 3\Delta t\, \omega^h)}$ → find this

minimize error between and actual numerical solution at the Gauss pts

$a_i = \pm c$ values at Gauss pts

we obtain a_0 ✓

3.3. Best-fit approach

Instead of evaluating dispersion and dissipation errors at one individual point on the outflow at the final time step, we propose a best-fit method to normalize errors for all points on the outflow face. We use a least square method to define the residual R between exact and numerical solutions as follows

$$R^2 = \sum_{I=1}^{ngp} |T^h(x_I) - e^{i(Kx_I - \omega^h t)}|^2, \tag{26}$$

where x_I indicates the location of Gauss points on the outflow face of a patch. Expanding Eq. 26 gives rise to the form of

$$R^2 = \sum_{I=1}^{ngp} |e^{iKx_I}|\,|e^{-iKx_I}T^h(x_I) - e^{-i\omega_r t}e^{\omega_i t}|^2. \tag{27}$$

$$\sum_{I=1} |(A_I - M) - i(N_I - B)|^2 \tag{28}$$

where

$$A_I = \text{Real}(e^{-iKx_I}T^h(x_I)) \tag{29a}$$

$$B_I = \text{Imag}(e^{-iKx_I}T^h(x_I)), \tag{29b}$$

and

$$e^{-i\omega_r t}e^{\omega_i t} = M + iN \tag{30}$$

Variables M and N in Eq. 30 are determined by the fact that the residual R^2 is extremum if $\frac{\partial R^2}{\partial M} = 0$ and $\frac{\partial R^2}{\partial N} = 0$. This leads to

$$M = \frac{\sum A_I}{ngp} \tag{31a}$$

$$N = \frac{\sum B_I}{ngp}. \tag{31b}$$

Letting $(Re^{i\phi})^{-1} = M + iN$, Eq. (30) can be expressed as

$$e^{-i\omega_r^h t}e^{\omega_i t} = (Re^{i\phi})^{-1}, \tag{32}$$

which gives

$$\omega_i^h = \frac{\log(1/R)}{t} \tag{33a}$$

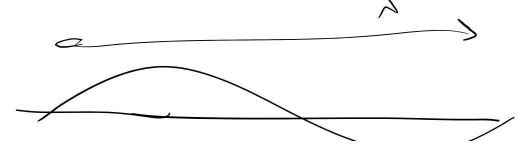
$$\omega_r^h = \frac{\phi + 2\pi m}{t} \tag{33b}$$

Results Stability analysis

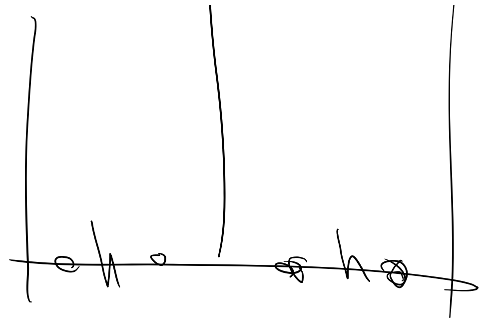
$$e^{i(Kx - \omega^h t)} = e^{i(Kx - \omega_r^h t)} \, e^{\omega_i^h t}$$

$$\boxed{\omega_i^h (K\Delta x) < 0} \quad (*)$$

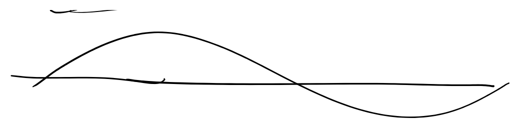
as $\Delta t \nearrow$ (*) is going to be violated



λ



$\lambda \ll h$



$K\lambda = 2\pi$

$\lambda = \frac{2\pi}{K}$

for stability analysis

$\lambda_{h(p)} = \frac{h}{p+1}$

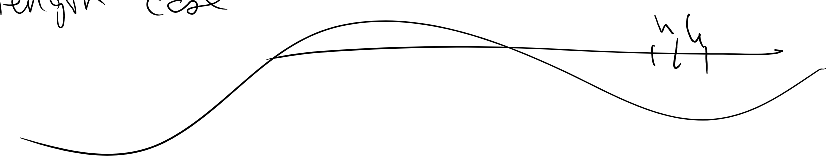
polynomial order

$\lambda = \frac{2\pi}{K}$

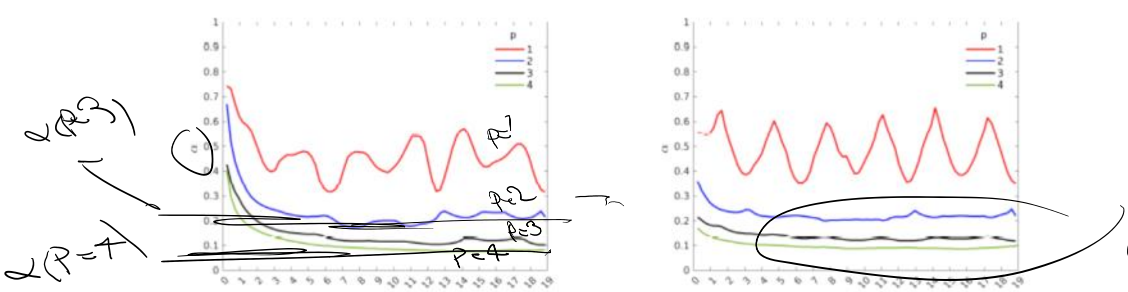
$\frac{\lambda}{\lambda_{h(p)}} = \frac{2\pi(p+1)}{Kh} = \frac{2\pi(p+1)}{\Theta} \geq 1$

$\boxed{\Theta \ll 2\pi(p+1) \quad \Theta = Kh}$

$\Theta \to 0$ wave long length case



$\Delta t = \alpha \frac{h^2}{D}$ $D = \frac{K}{C}$ for parabolic case



(a) Parabolic equation. 5 layers

(b) Parabolic equation. 15 layers

as $p \nearrow$ $(K\lambda)$

α stabilizes