Partial Differential Equations (PDEs) classification groups

PDEs can be classified from different perspectives:

1. **Order of PDE**: The highest order of PDE

$u_t = u_{xx}$ (second order)

$u_t = u_x$ (first order)

$u_t = uu_{xxx} + \sin \square$ (third order)

- $u_t + u_{xxx} + uu_x = 0$ (KdV Eqn., third order)
- $u_x^2 + u_y^2 = c^2$ (Eikonal Eqn. of Geometric Optics, first order)

2. **Number of variables**: The number of independent variables for all the involved functions:

$u_t = u_{xx}$ (two variables:

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \quad \text{(three variables: } r, \theta, \text{ and } t)$$

Note: Under certain conditions when initial / boundary conditions and partial derivatives are independent of a given variable we can reduce the number of variables.

Source: [Farlow, 2012]

Partial Differential Equations (PDEs) classification groups

3. Homogeneity: If the source term (right hand side) of the equation is zero the PDE is called homogeneous. The same concept applies to initial (IC) and boundary (BC) conditions of a PDE (RHS of the IC/BC differential operator is zero)

- $u_t + u_x = 0$ is homogeneous
- $u_{xx} + u_{yy} = x^2 + y^2$ is inhomogeneous

4. Type of coefficient:

- Constant coefficient (function & its derivative terms have constant coefficients)

$$3\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2}x + 2\frac{\partial u}{\partial y} = 0$$

- Variable coefficient
  - Coefficients only function of independent variables (e.g. x, t)

$$e^{2x}\frac{\partial^2 u}{\partial x^2} + 2\,e^{x+y}\frac{\partial^2 u}{\partial x \partial y} + e^{2y}\frac{\partial^2 u}{\partial y^2} = 0.$$

  - Coefficients function of independent variables AND the function (e.g. x, t, u)

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + u^2 = 0$$

Source: [Farlow, 2012]

Partial Differential Equations (PDEs) classification groups

5. **Hyperbolic / parabolic / elliptic PDEs**:
   - The classification becomes more clear in the next few slides. Below is the brief description of some of their characteristics and sample applications.
   - **Hyperbolic PDEs** correspond to the propagation of waves and there is a finite speed of propagation of waves. They tend to preserve or generate discontinuities (in the absence of damping). Hyperbolic PDEs are often transient although some steady-state limits of transient PDEs can be hyperbolic as well (e.g. steady advection problems).
     Examples: Elastodynamics, Transient electromagnetics; Acoustic equation.
   - **Parabolic PDEs**: Unlike hyperbolic PDEs the speed of propagation of information is infinite for parabolic PDEs. They also tend to dissipate sharp solution features and have a "diffusive" behavior. Many transient diffusion problems are modeled (or idealized) by parabolic PDEs. Some examples are
     Examples: Fourier heat equation; Viscous flow (Navier-Stokes equations)
   - **Elliptic PDEs**: Elliptic problems are characterized by the global coupling of the solution. They often correspond to steady-state limit of hyperbolic and parabolic PDEs.

Source: [Farlow, 2012]

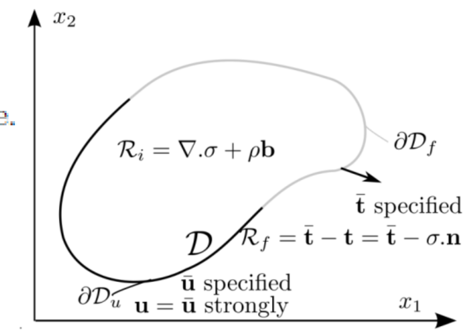
Types of PDEs
Elliptic, parabolic, hyperbolic

1. **Elliptic equations**:

- Sample:

$$\nabla.\sigma + \rho\mathbf{b} = 0 \qquad \text{Elastostatics equation}$$

$$\Delta u = f, \Delta u = \Sigma_{i=1}^{d} u_{,ii} \text{ (Laplacian)}, \qquad \text{Poisson equation}$$



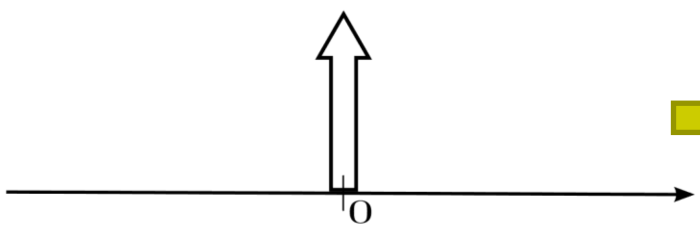
- The entire domain is physically coupled and often numerical methods involve a global solve.
- They are often steady state limit of parabolic/hyperbolic systems.

2. **Parabolic equations**(dynamic):

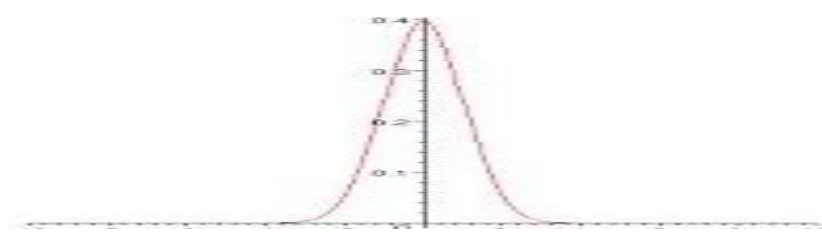
- Sample:

$$C\frac{\mathrm{d}T}{\mathrm{d}t} - \kappa\Delta T = Q, \text{(constant } \kappa\text{)}, \qquad \text{Parabolic(Fickian) heat equation}$$

- Imply an infinite speed of propagation of information.
- The entire spatial domain is coupled.
- Numerical methods may involve the solution of the global spatial domain or local domains.
- The diffusive operator **smoothens the solution**.



$$T(x, t = 0) = \delta(x)$$



$$G(x,t) = \sqrt{\frac{C}{4\pi\kappa t}}\exp\left(-\frac{Cx^2}{4\kappa t}\right) \quad \text{Green's function}$$

Types of PDEs
Elliptic, parabolic, hyperbolic

3. **Hyperbolic equations**(dynamic):

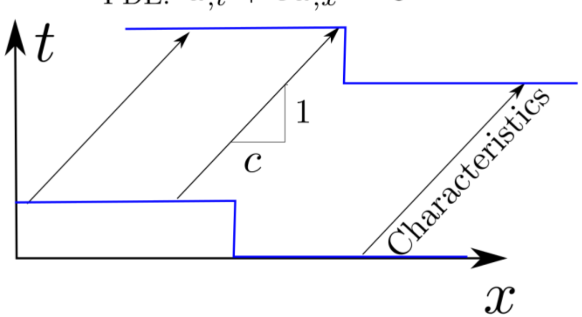
- Sample:

$$\rho \frac{\mathrm{d}^2 \mathbf{u}}{\mathrm{d}t^2} - \nabla . \sigma = \rho \mathrm{b} \qquad \text{Elastostatics equation}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - k \Delta u = f$$

- There is a maximum speed for the propagation of waves (information).
- Due to finiteness of the wave speed the spatial domain is NOT globally coupled.
- Numerical methods may employ the locality of hyperbolic systems to devise local solution schemes.
- Unlike parabolic equations, hyperbolic equations preserve discontinuities or even generate them (nonlinear equations).

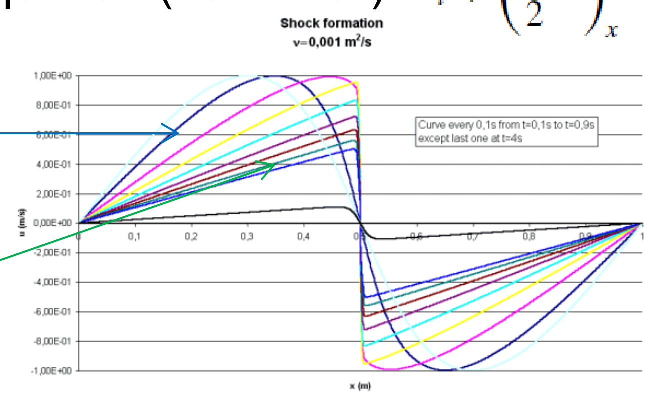
PDE: $u_{,t} + c u_{,x} = 0$



Burger's equation (nonlinear) $u_t + \left(\frac{1}{2}u^2\right)_x = 0$

t = 0, smooth solution

t > 0, shock has formed



Partial Differential Equations (PDEs) classification groups

6. Linearity:

- The PDE is linear if the dependent variable and all its derivatives appear linearly in the PDE. The nonlinear PDEs are classified into several groups as their solution characteristics can be quite distinct:

Notations:

- Multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$

  order of the multi-index is $|\alpha| = \alpha_1 + \ldots + \alpha_n$

- Multi-index partial derivative

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u.$$

  example: $\alpha = (1, 2)$ ⇒ $D^\alpha u = \partial_x \partial_y^2 u = u_{xyy}$

- Collection of all partial derivatives of order *k*: $D^k u = \{D^\alpha u \; : \; |\alpha| = k\}$

  example: If $u = u(x_1, \ldots, x_n)$, then $D^1 u = \{u_{x_i} \; : \; i = 1, \ldots, n\}$.

Source: [Levandosky, 2002]

Partial Differential Equations (PDEs) llinear/nonlinear classifications

General form of PDE:

$$F(\vec{x}, u, Du, D^2u, \ldots, D^ku) = 0$$

Linear → **nonlinear** (in order)

A. Linear: If u and its derivatives appear in a linear fashion. That is F can be written as,

$$\sum_{|\alpha|\leq k} a_\alpha(\vec{x}) D^\alpha u = f(\vec{x}).$$

Examples:

- $u_t + u_x = 0$ is homogeneous linear
- $u_{xx} + u_{yy} = 0$ is homogeneous linear.
- $u_{xx} + u_{yy} = x^2 + y^2$ is inhomogeneous linear.
- $u_t + x^2 u_x = 0$ is homogeneous linear.
- $u_t + u_{xxx} + uu_x = 0$ is not linear.
- $u_x^2 + u_y^2 = 1$ is not linear.

Source: [Levandosky, 2002]

Partial Differential Equations (PDEs) llinear/nonlinear classifications

B. Semi-linear: is a nonlinear PDE where the highest order derivatives can be written in a linear fashion of functions of ***x***. That is, such coefficients are only functions of independent coordinate ***x***. The PDE can be written as,

$$\sum_{|\alpha|=k} a_\alpha(\vec{x}) D^\alpha u + a_0(D^{k-1}u, D^{k-2}u, \ldots, Du, u, \vec{x}) = 0.$$

Examples:

- $u_t + u_{xxx} + uu_x = 0$ is semilinear.
- $u_{xx} + u_{yy} = u^3$ is semilinear.
- $u_t + xu_x = 0$ is linear.
- $u_t + uu_x = 0$ quasilinear butnot semilinear.

Source: [Levandosky, 2002]

Partial Differential Equations (PDEs) llinear/nonlinear classifications

C. Quasi-linear: is a nonlinear PDE, that is not semilinear and its highest derivatives can be written as linear function of functions of x **and lower order derivatives of u**. That is, it can be written as,

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u,\ldots,Du,u,\vec{x})D^\alpha u + a_0(D^{k-1}u,\ldots,Du,u,\vec{x}) = 0.$$

That is the coefficients of highest order terms depend on

$\vec{x}, u, \ldots, D^{k-1}u,$ but not on $D^k u.$

Examples:

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$ Is

quasilinear
semilinear if a(x, y), b(x,y)
linear if a(x, y), b(x, y) and c = u d(x, y)

- $u_t + a(u)u_x = 0$ is quasilinear.
- $u_x^2 + u_y^2 = 1$ is not quasilinear.

Source: [Levandosky, 2002]

Partial Differential Equations (PDEs) llinear/nonlinear classifications

D. Fully-nonlinear: If it's nonlinear and cannot be written in quasi-linear, semi-linear forms.

$$\frac{\partial u}{\partial x_1} + \left(\frac{\partial u}{\partial x_2}\right)^2 = 0$$

$$u_{xx}\, u_{yy} - (u_{xy})^2 = \psi \qquad \text{Monge-Ampère equation}$$

- $u_x^2 + u_y^2 = c^2$ (Eikonal Eqn. of Geometric Optics, first order)

For a list of well-known nonlinear (semi-linear, quasi-linear, and fully nonlinear) PDEs refer to here (https://en.wikipedia.org/wiki/List_of_nonlinear_partial_differential_equations)

Source: [Levandosky, 2002]

Partial Differential Equations (PDEs) classification groups

Linearity	Linear				Nonlinear		
Order	1	2	3	4	5	. . .	*m*
Kinds of coefficients (linear equations)	Constant				Variable		
Homogeneity (linear equations)	Homogeneous				Nonhomogeneous		
Number of variables	1	2	3	4	5	. . .	*n*
Basic type (linear equations)	Hyperbolic			Parabolic		Elliptic	

FIGURE 1.1 Classification diagram for partial differential equations.

Nonlinear →

- Semi-linear
- Quasi-linear
- Fully-nonlinear

Source: [Farlow, 2012]

Solution to 1D wave equation

- We are interested in solving the **1st order linear PDE in two variables**:

$$\text{PDE} \quad a(x,t)u_x + b(x,t)u_t + c(x,t)u = 0 \qquad -\infty < x < \infty$$
$$0 < t < \infty$$
$$\text{IC} \quad u(x,0) = \phi(x) \qquad -\infty < x < \infty$$

  - The idea is to express the two partial derivatives as one derivative term:

$$a(x,t)u_x + b(x,t)u_t \longrightarrow \frac{du}{ds}$$

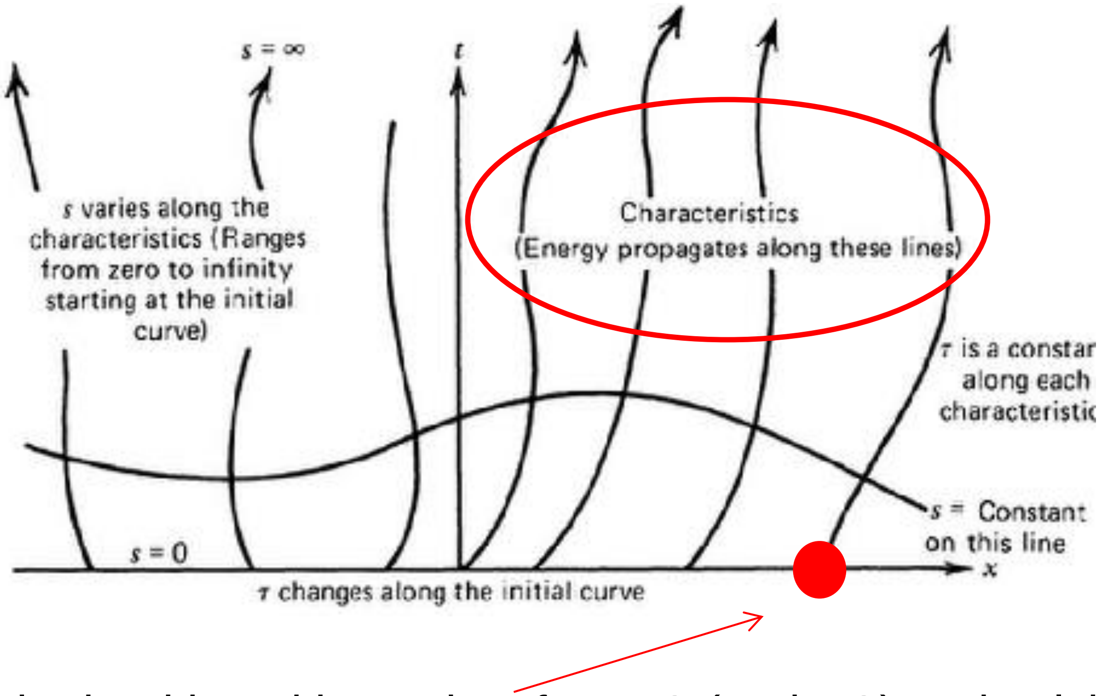
  - Thus the **PDE turns to an ODE**:

$$\frac{du}{ds} + c(x,t)u = 0$$

Source: [Farlow, 2012]

Solution to 1D wave equation

- The ODEs are solved along characteristic where variable s changes



- Solution is obtained by taking points from IC (and BC) and solving for the solution along the characteristic

Source: [Farlow, 2012]

Solution to 1D wave equation

- How this is done? By using the chain rule:

$$\frac{du}{ds} = u_x \frac{dx}{ds} + u_t \frac{dt}{ds} = a(x,t)u_x + b(x,t)u_t$$

$$\frac{du}{ds} + c(x,t)u = 0$$

where x and t are obtained from ODEs:

$$\frac{dx}{ds} = a(x,t)$$

$$\frac{dt}{ds} = b(x,t)$$

- The solution to ODEs ★ provides the direction of characteristics

Source: [Farlow, 2012]

Solution to 1D wave equation
Example: constant coefficients

- Consider the following PDE with constant coefficients:

$$\text{PDE} \quad u_x + u_t + 2u = 0 \qquad -\infty < x < \infty \qquad 0 < t < \infty$$
$$\text{IC} \quad u(x,0) = \sin x \qquad -\infty < x < \infty$$

- The characteristic equations become:

$$\frac{dx}{ds} = a(x,t) \qquad \frac{dt}{ds} = b(x,t) \quad \Rightarrow \quad \frac{dx}{ds} = 1 \qquad \frac{dt}{ds} = 1 \qquad 0 < s < \infty \quad \Rightarrow$$

$$x(s) = s + c_1 \qquad t(s) = s + c_2$$

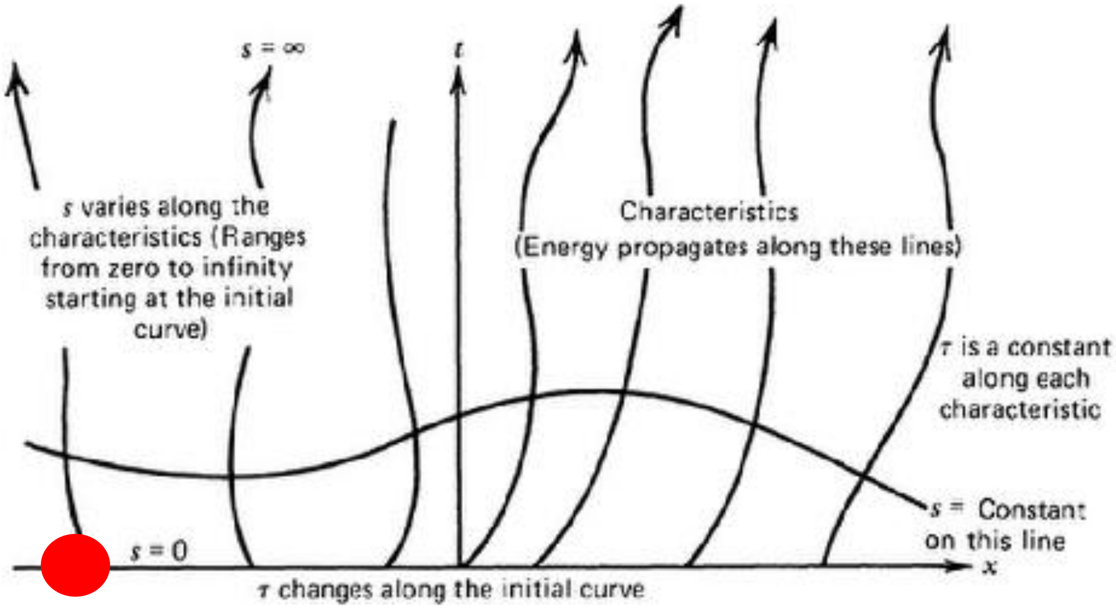
- c_1 and c_2 can be obtained from initial condition

$x(0) = \tau$ - At the start of characteristic lines x is equal to a secondary variable τ (it can also be set to x)

$t(0) = 0$ - Time is equal to zero at the beginning of characteristic lines

Source: [Farlow, 2012]

Solution to 1D wave equation
Example: constant coefficients



$t(0) = 0$

Time is equal to zero at the beginning of characteristic lines

$x(0) = \tau$

Note: We can simply write x(0) = x and not introduce the intermediate parameter τ.

Source: [Farlow, 2012]

Solution to 1D wave equation
Example: constant coefficients

- Continuation, the solution of PDE

$$\text{PDE} \quad u_x + u_t + 2u = 0 \qquad -\infty < x < \infty \qquad 0 < t < \infty$$

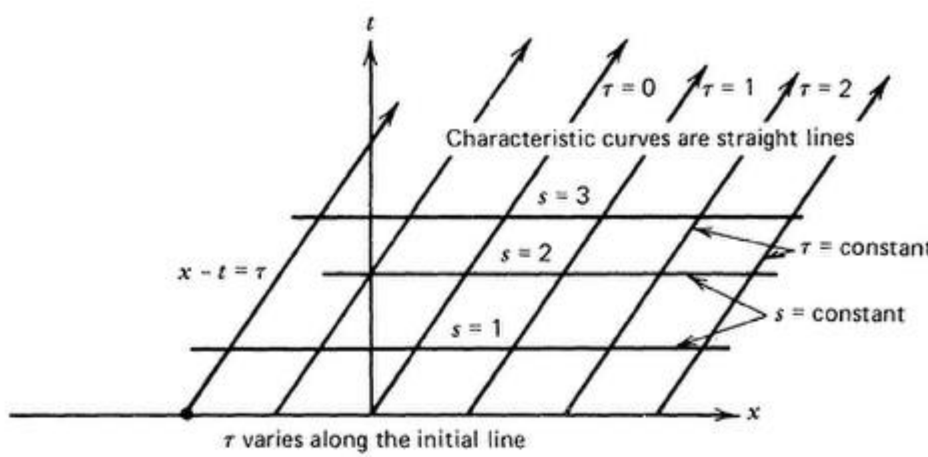
$$\text{IC} \quad u(x,0) = \sin x \qquad -\infty < x < \infty$$

- And characteristic equations $x(s) = s + c_1 \qquad t(s) = s + c_2$

  with $x(0) = \tau$, $t(0) = 0$ follow as:

$$x = s + \tau$$
$$t = s$$

- We note that $x - t = \tau$ and characteristics look like



Source: [Farlow, 2012]

Solution to 1D wave equation
Example: constant coefficients

- Thus the PDE turns to the following ODE:

$$\text{ODE} \qquad \frac{du}{ds} + 2u = 0 \qquad 0 < s < \infty$$

$$\text{IC} \qquad u(0) = \sin \tau$$

- Solving the *Initial Value Problem* (IVP) we get

$$u(s,\tau) = \sin \tau \, e^{-2s} \quad \triangle$$

- and by inverting (s, τ) → (x, t) $\quad x = s + \tau \qquad t = s$
  we get
  $s = t \qquad \tau = x - t$

- Thus △ becomes $\quad \boxed{u(x,t) = \sin(x - t)e^{-2t}}$

Source: [Farlow, 2012]

Solution to 1D wave equation
Example: constant coefficients

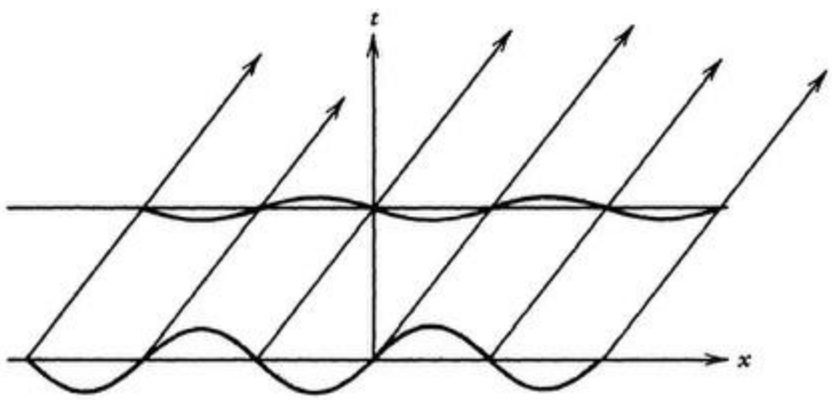
- So the solution to

$$\text{PDE} \quad u_x + u_t + 2u = 0 \quad -\infty < x < \infty \quad 0 < t < \infty$$

$$\text{IC} \quad u(x,0) = \sin x \quad -\infty < x < \infty$$

Is

$$u(x,t) = \sin(x - t)e^{-2t}$$



Which is a sine wave moving with speed 1 and damping in time

Source: [Farlow, 2012]

Solution to 1D wave equation
Example: variable coefficients

- Consider the problem

$$\text{PDE} \quad xu_x + u_t + tu = 0 \qquad -\infty < x < \infty \qquad 0 < t < \infty$$
$$\text{IC} \quad u(x,0) = F(x) \qquad \text{(an } \textit{arbitrary} \text{ initial wave)}$$

- Step 1: The ODEs for characteristics are:

$$\frac{dx}{ds} = x \text{ has solution} \qquad x(s) = c_1 e^s$$

$$\frac{dt}{ds} = 1 \text{ has solution} \qquad t(s) = s + c_2$$

Letting $x(0) = \tau$ and $t(0) = 0$ gives $c_1 = \tau$ and $c_2 = 0$.

Hence

$$x = \tau e^s$$
$$t = s$$

Source: [Farlow, 2012]

Solution to 1D wave equation
Example: variable coefficients

- Step 2: we solve the ODE $\frac{du}{ds} + su = 0 \qquad 0 < s < \infty$

  we IC $u(o) = F(\tau)$

- Which gives $u(s,\tau) = F(\tau)e^{-s^2/2}$ ★

- Step 3: By inverting $x = \tau e^{s}$

  $t = s$

  we get

  $s = t, \; \tau = xe^{-t}$,

  and by plugging back into ★ we get the final solution

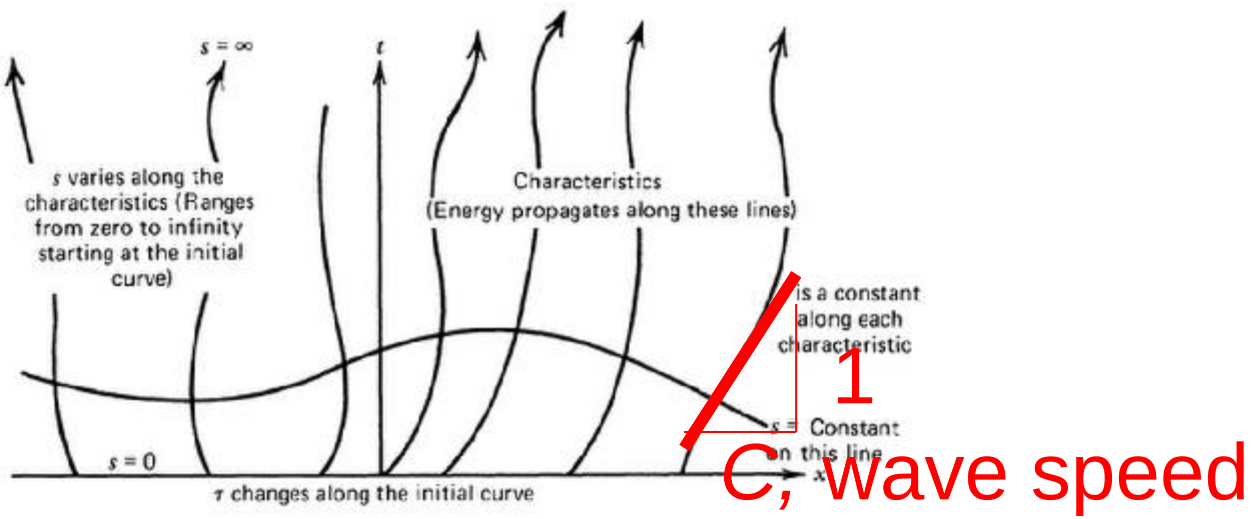
$$u(x,t) = F(xe^{-t})e^{-t^2/2}$$

e.g. for initial condition F(x) = sin(x) we have $u(x,t) = \sin\,(xe^{-t})e^{-t^2/2}$

Source: [Farlow, 2012]

Solution to 1D wave equation
Discussion on hyperbolicity

- Key aspect of the first order PDEs we discussed was the solution for characteristic curves along which the solution could be obtained by the solution of an ODE.
  The spatial to temporal slope of characteristics corresponds to the wave speed.



- The hyperbolicity of a PDE corresponds to having characteristic curves along which the solution propagates. For higher order PDEs we investigate if we can breakdown the PDEs into the solution of ODEs along characteristic curves. If this is possible the PDE is hyperbolic and has a finite speed of information propagation at a given point.

Classification of second order PDEs: Two independent parameters

- Consider a general 2nd order PDE

$$F(\vec{x}, u, Du, D^2u) = 0.$$

- We restrict our attention to linear PDE with 2 independent parameters below (results can easily be generalized to semi-linear case):

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

where A, B, C, D, E, F, G are functions of (x, y) in general (linear PDE). The classification of PDE at a given point (x_0, y_0) is as follows:

1. Hyperbolic at a point (x_o,y_o) if $B^2(x_o,y_o) - 4A(x_o,y_o)C(x_o,y_o) > 0$.
2. Parabolic at a point (x_o,y_o) if $B^2(x_o,y_o) - 4A(x_o(x_o,y_o)C(x_o,y_o) = 0$.
3. Elliptic at a point (x_o,y_o) if $B^2(x_o,y_o) - 4A(x_o,y_o)C(x_o,y_o) < 0$.

- If 1 holds for all (x, y) the PDE is called hyperbolic for all positions (same for 2 and 3)

Classification of second order PDEs: Two independent parameters

- Examples:

- The wave equation

$$u_{tt} - u_{xx} = 0 \qquad \text{is hyperbolic.}$$

A = 1, B = 0, C = -1 ⇨
$B^2 – 4AC = 4 > 0$

- The Laplace equation

$$u_{xx} + u_{yy} = 0 \qquad \text{is elliptic.}$$

A = 1, B = 0, C = 1 ⇨
$B^2 – 4AC = -4 < 0$

- The heat equation

$$u_t - u_{xx} = 0 \qquad \text{is parabolic.}$$

A = 0, B = 0, C = -1 ⇨
$B^2 – 4AC = 0$

Classification of second order PDEs: Canonical form

- The idea is to cast the PDE in the canonical form

1. $$\begin{cases} u_{\xi\xi} - u_{\eta\eta} = \Psi(\xi, \eta, u, u_\xi, u_\eta) \\ u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta) \end{cases} \quad \begin{pmatrix} \text{two canonical forms for} \\ \text{the hyperbolic equation} \end{pmatrix}$$

2. $$u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta) \quad \begin{pmatrix} \text{the canonical form for} \\ \text{the parabolic equation} \end{pmatrix}$$

3. $$u_{\xi\xi} + u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta) \quad \begin{pmatrix} \text{the canonical form for} \\ \text{the elliptic equation} \end{pmatrix}$$

Source: [Farlow, 2012]

Classification of second order PDEs: Canonical form

- We look for parameters ξ and η that cast the PDE into the hyperbolic form

$$u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$$

- By transformation

$$\xi = \xi(x,y)$$
$$\eta = \eta(x,y)$$

- By change of parameters we obtain

$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy} \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}
\end{aligned}$$

Source: [Farlow, 2012]

Classification of second order PDEs: Canonical form

- Substituting into the original PDE $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ we obtain

$$\overline{A}u_{\xi\xi} + \overline{B}u_{\xi\eta} + \overline{C}u_{\eta\eta} + \overline{D}u_{\xi} + \overline{E}u_{\eta} + \overline{F}u = \overline{G}$$

where

$$\begin{aligned}
\overline{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
\overline{B} &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
\overline{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
\overline{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
\overline{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
\overline{F} &= F \\
\overline{G} &= G
\end{aligned}$$

- To cast it in the form $u_{\xi\eta} = \Phi(\xi, \eta, u, u_{\xi}, u_{\eta})$ we need to set $\overline{A}$ and $\overline{C}$ to zero.

$$\begin{aligned}
\overline{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \\
\overline{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0
\end{aligned}$$

Source: [Farlow, 2012]

Classification of second order PDEs: Canonical form

- Which results in equations of the form

$$A[\xi_x/\xi_y]^2 + B[\xi_x/\xi_y] + C = 0$$
$$A[\eta_x/\eta_y]^2 + B[\eta_x/\eta_y] + C = 0$$

- The solutions to these equations are:

$$[\xi_x/\xi_y] = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

(characteristic equations)

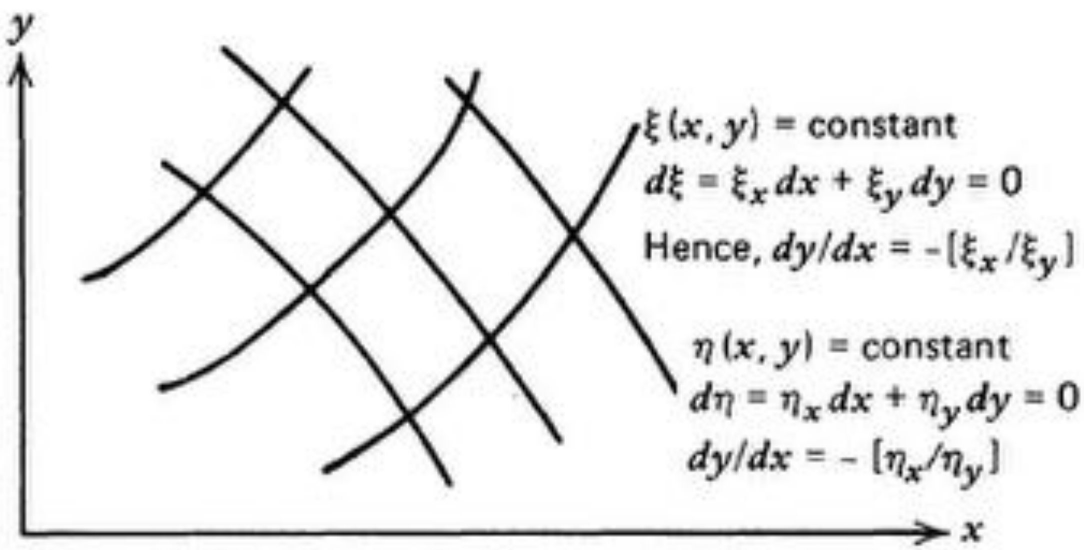
$$[\eta_x/\eta_y] = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

- He have three cases:
  - $B^2 - 4AC > 0$: Two distinct values for $\xi_x/\ \xi_y$ and $\eta_x/\ \eta_y$. We can cast the equation in hyperbolic canonical form.
  - $B^2 - 4AC = 0$: ONLY one distinct values for $\xi_x/\ \xi_y$ and $\eta_x/\ \eta_y$. We cannot cast the equation in hyperbolic form, but can cast in parabolic form.
  - $B^2 - 4AC < 0$: NO REAL roots $\xi_x/\ \xi_y$ and $\eta_x/\ \eta_y$. We cannot cast the equation in hyperbolic canonical form, but can cast in elliptic form.

Classification of second order PDEs: Characteristic values and curves

- ξ and η are called the characteristic parameters (similar to the first ordr PDE)

- By solving the previous page 2nd order equation we can find how the contour lines (constant values) for ξ and η look like in (x, y) space.



Classification of second order PDEs: Example

- A constant coefficient hyperbolic example:

$$u_{xx} - 4u_{yy} + u_x = 0$$

- The equations for characteristic curves are:

$$\frac{dy}{dx} = -[\xi_x/\xi_y] = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -2$$

$$\frac{dy}{dx} = -[\eta_x/\eta_y] = \frac{B + \sqrt{B^2 - 4AC}}{2A} = 2$$

- After integration we obtain $y = -2x + c_1$

$$y = 2x + c_2$$

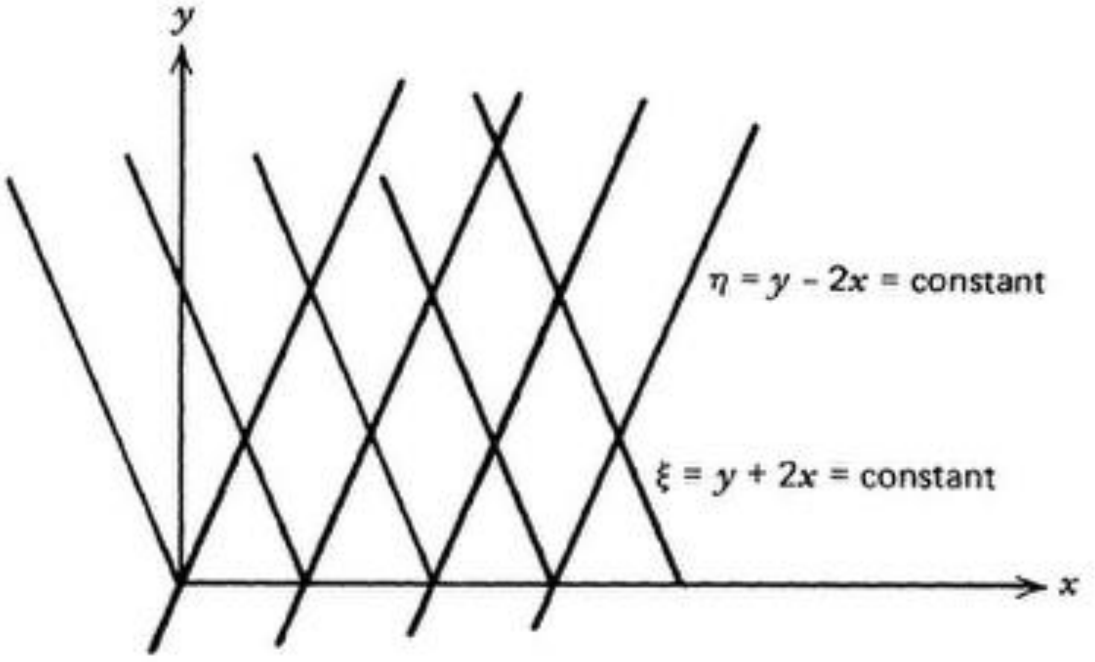
- By leaving x and y on the RHS of equation we obtain equations for ξ and η:

$$\xi = y + 2x = c_1$$

$$\eta = y - 2x = c_2$$

Classification of second order PDEs: Example

- And the characteristic curves look like



Classification of second order PDEs: Example with variable coefficients

- Consider the PDE

$$y^2 u_{xx} - x^2 u_{yy} = 0 \quad x > 0 \quad y > 0$$

which is a hyperbolic equation in the first quadrant.

- We find the characteristics by the equation

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = -\frac{x}{y}$$

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{x}{y}$$

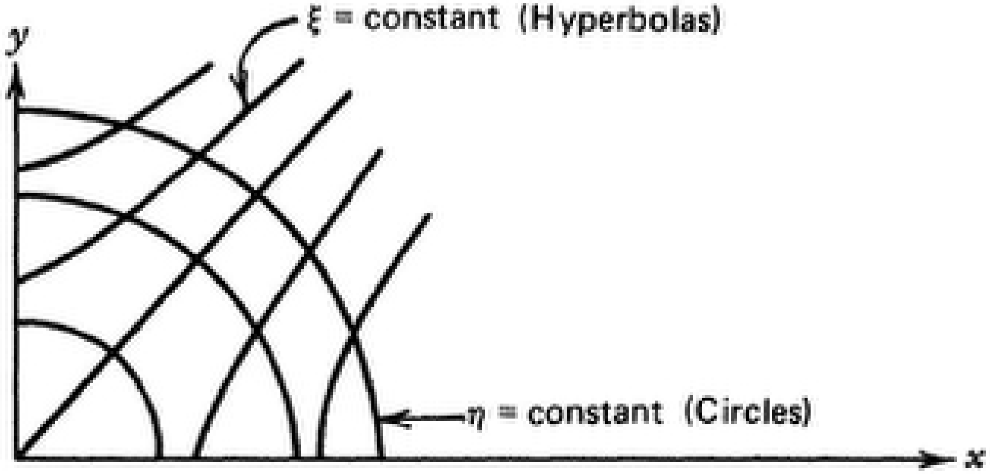
- By solving these equations and moving x, y to the RHS we obtain

$$y^2 - x^2 = \text{constant}$$
$$y^2 + x^2 = \text{constant}$$

$\Rightarrow$

$$\xi = y^2 - x^2$$
$$\eta = y^2 + x^2$$

Classification of second order PDEs: Example with variable coefficients



- To obtain the form of the equation in the canonical form

$$\bar{A}u_{\xi\xi} + \bar{B}u_{\xi\eta} + \bar{C}u_{\eta\eta} + \bar{D}u_{\xi} + \bar{E}u_{\eta} + \bar{F}u = \bar{G}$$

we compute

$$\begin{aligned}
\bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
\bar{B} &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
\bar{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
\bar{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
\bar{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
\bar{F} &= F \\
\bar{G} &= G
\end{aligned}$$

where $\bar{A}$ and $\bar{C}$ are zero (why?)

Classification of second order PDEs: Example with variable coefficients

- To obtain

$$u_{\xi\eta} = \frac{-(x^2 + y^2)u_\xi + (y^2 - x^2)u_\xi}{8x^2y^2}$$

- And by solving (x, y) in terms of ξ and η we obtain:

$$u_{\xi\eta} = \frac{\eta u_\xi - \xi u_\eta}{2(\xi^2 - \eta^2)}$$

Classification of second order PDEs: Summary of canonical forms

- For a second order hyperbolic PDE in the form

$$u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta)$$

Hyperbolic 1st canonical form

- by the change of parameter

$$\alpha = \alpha(\xi,\eta) = \xi + \eta$$
$$\beta = \beta(\xi,\eta) = \xi - \eta$$

- We cast it into the 2nd canonical form:

$$u_{\alpha\alpha} - u_{\beta\beta} = \psi(\alpha, \beta, u, u_\alpha, u_\beta)$$

Hyperbolic 2nd canonical form

Note: In fact for elliptic PDEs by the same form of transformation we can cast the PDE into elliptic canonical form:

$$u_{\alpha\alpha} + u_{\beta\beta} = \psi(\alpha, \beta, u, u_\alpha, u_\beta)$$

Elliptic canonical form

For the derivation of this form and the parabolic canonical form refer to lesson 41 of [Farlow, 2012]:

$$u_{\alpha\alpha} = \psi(\alpha, \beta, u, u_\alpha, u_\beta)$$

Parabolic canonical form

Classification of second order PDEs: More than 2 independent variables

- For a second order linear hyperbolic PDE with n independent variables:

$$\sum_{i,j=1}^{n} a_{ij}\,\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} b_i\,\frac{\partial u}{\partial x_i} + c\,u + g = 0\,.$$

Note **a** is expressed as **symmetric matrix**

The classification is as follows:

- (H) for ($Z = 0$ and $P = 1$) or ($Z = 0$ and $P = n - 1$)
- (P) for $Z > 0$ ($\Leftrightarrow \det \mathbf{a} = 0$)
- (E) for ($Z = 0$ and $P = n$) or ($Z = 0$ and $P = 0$)
- (ultraH) for ($Z = 0$ and $1 < P < n - 1$)

where

- Z: nb. of zero eigenvalues of $\mathbf{a}$
- P: nb. of strictly positive eigenvalues of $\mathbf{a}$

The alternatives in the (H) and (P) definitions are due to the fact that multiplication by -1 of the equation leaves it unchanged.

Source: [Loret, 2008]

Classification of second order PDEs: More than 2 independent variables

Canonical form after coordinate transformation (refer to Loret chapter 3)

- Elliptic:

$$\sum_{i=1}^{n} u_{x_i x_i} + \ldots = 0.$$

$$\mathbf{a} = \mathrm{diag}(1, 1, \cdots, 1) \quad \Rightarrow$$
$$\lambda_i = 1 \quad (i \leq n)$$

- Hyperbolic:

$$u_{x_1 x_1} - \sum_{i=2}^{n} u_{x_i x_i} + \ldots = 0.$$

$$\mathbf{a} = \mathrm{diag}(-1, 1, \cdots, 1) \quad \Rightarrow$$
$$\lambda_1 = -1, \lambda_i = 1 \quad (2 \leq i \leq n)$$

- Parabolic:

$$\sum_{i=2}^{n} u_{x_i x_i} + \ldots = 0.$$

$$\mathbf{a} = \mathrm{diag}(0, 1, \cdots, 1) \quad \Rightarrow$$
$$\lambda_1 = 0, \lambda_i = 1 \quad (2 \leq i \leq n)$$

Source: [Loret, 2008]

Classification of second order PDEs: More than 2 independent variables

Comparison with 2nd order PDE with two variables:

$$Au_{x_1x_1} + Bu_{x_1x_2} + Cu_{x_2x_2} + \text{lower order terms (LOTS)} = 0 \quad \Rightarrow$$

$$\begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \end{bmatrix} \mathbf{a} \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix} u + \text{LOTS} = 0$$

where

$$a = \begin{bmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{bmatrix}$$

has the eigenvalues derived from $\lambda^2 - (A+C)\lambda - \frac{1}{4}(B^2 - 4AC) = 0$

so if $\Delta = B^2 - 4AC$ is

$$\begin{cases} \Delta > 0 & \text{Two roots } \lambda_1\lambda_2 < 0 & \text{Hyperbolic PDE} \\ \Delta = 0 & \text{One zero root (not two otherwise PDE is first order)} & \text{Parabolic PDE} \\ \Delta < 0 & \text{Two roots } \lambda_1\lambda_2 > 0 & \text{Elliptic PDE} \end{cases}$$

D'Alembert solution of the wave equation

Goals:

- Obtain the solution to the wave equation
- Solution of the PDE using characteristics from the PDE's canonical form

PDE $\quad u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty \qquad 0 < t < \infty$

ICs $\quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \qquad -\infty < x < \infty$

The solution is,

$$u(x,t) = \frac{1}{2}\left[f(x - ct) + f(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

D'Alembert solution of the wave equation Solution using the PDE's canonical form

- The characteristic parameters

$$\xi = x + ct$$
$$\eta = x - ct$$

cast the PDE into its canonical form

$$u_{\xi\eta} = 0$$

Now we can **integrate the PDE on ξ and η respectively**.

2 ODEs!

$$u(\xi,\eta) = \Phi(\eta) + \psi(\xi)$$

D'Alembert solution of the wave equation
Use of initial conditions

- By plugging in the initial conditions we want to solve the functions ϕ and ψ:

$$\text{ICs} \quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \quad -\infty < x < \infty$$

$$u(\xi,\eta) = \Phi(\eta) + \psi(\xi)$$

$$\phi(x) + \psi(x) = f(x)$$
$$-c\phi'(x) + c\psi'(x) = g(x)$$

- By integrating the second equation we get

$$\phi(x) + \psi(x) = f(x)$$
$$-c\phi(x) + c\psi(x) = \int_{x_0}^{x} g(\xi)\, d\xi + K$$

$$\phi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\xi)\, d\xi$$
$$\psi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\xi)\, d\xi$$

(K is set to zero because eventually in the solution of u K cancels out)

D'Alembert solution of the wave equation
Final solution, left- and right- going waves

$$u(\xi,\eta) = \Phi(\eta) + \psi(\xi) = \varphi(x - ct) + \psi(x + ct)$$

Right-going wave (speed c)

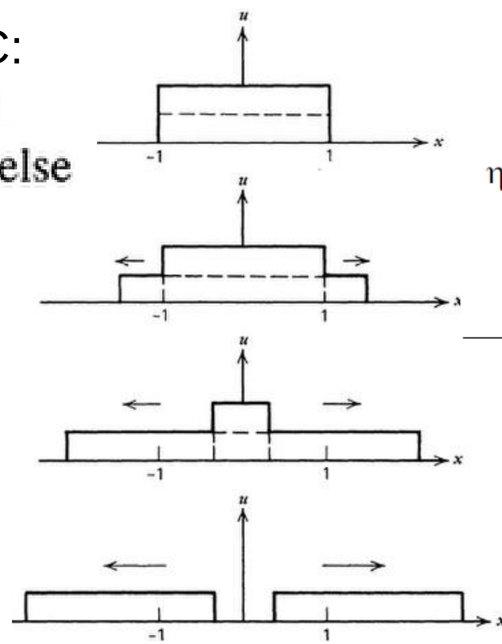
Left-going wave (speed c)

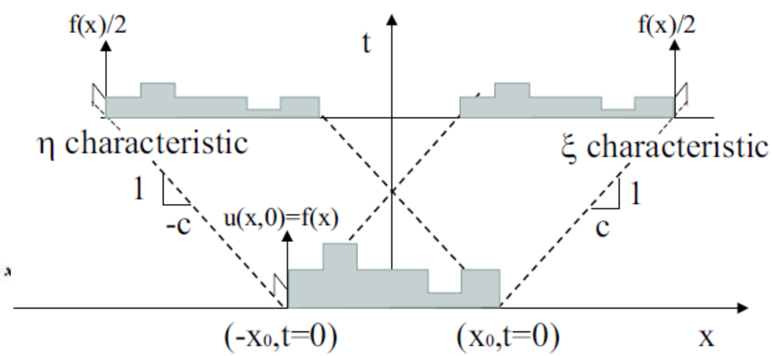
$$u(x,t) = \frac{1}{2}\left[f(x - ct) + f(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

Example: for the following IC:

$$u(x,0) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{everywhere else} \end{cases}$$

$$u_t(x,0) = 0$$

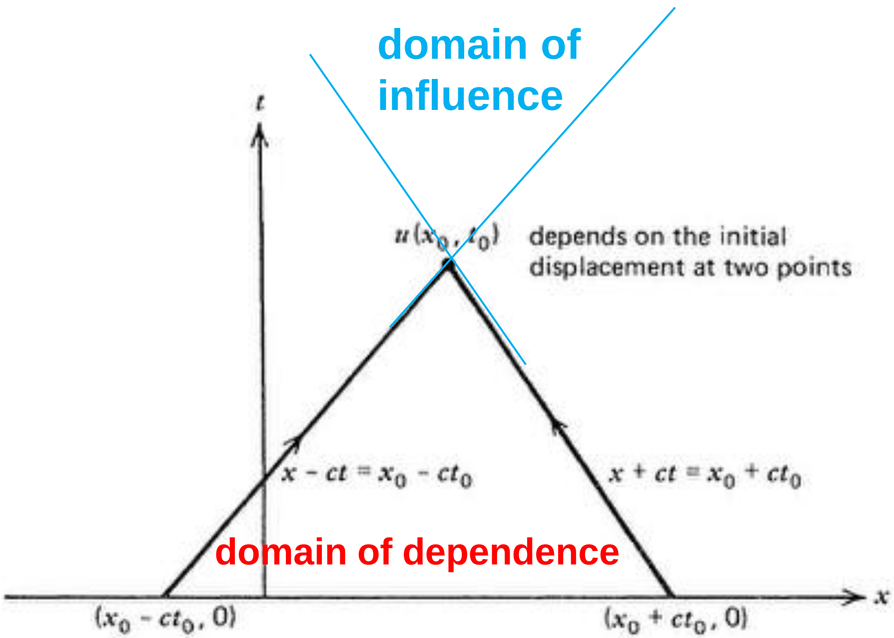


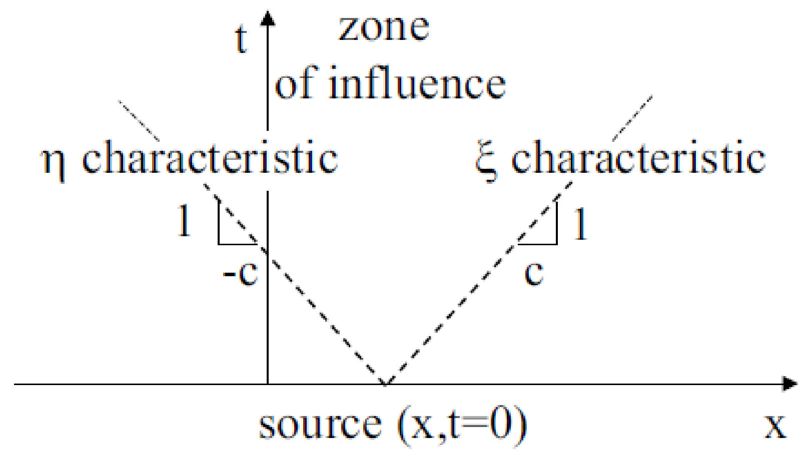


The initial displacement is halved and propagated to the left and right with speeds c

Domain of influence and dependence
Finite speed of information propagation

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi$$





Solution at x_0, t_0 only depends on the IC in $[x_0 - c\,t_0, x_0 + c\,t_0]$ which is called **domain of dependence**

Region where the solution of (x_0, t_0) influences is called **domain of influence**

Systems of 1st order PDEs (conservation laws) in 1D (2 independent parameters)

- Assume we want to solve the system of semi-linear first order PDEs,

$$\text{PDE}: \quad \mathbf{q}_{,t} + \mathbf{A}\mathbf{q}_{,x} = \mathbf{s}(\mathbf{q}, x, t) \quad (1a)$$

$$\text{IC}: \quad \mathbf{q}(x, 0) = \mathbf{q}_0(x) \quad (1b)$$

where

$$\mathbf{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad \text{vector of unknown fields}$$

$$\mathbf{A} \quad n \times n \text{ flux matrix}$$

$$\mathbf{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \quad \text{source term (can be nonlinear in } \mathbf{q}$$

$$n \quad \text{number of fields}$$

Systems of 1st order PDEs
Characteristic values

- While we know how to solve ONE first order PDE, we cannot solve this system because the PDEs are coupled through the arbitrary matrix $\mathbf{A}$
- If we have matrices as follows,

$$\mathbf{L}\mathbf{A} = \mathbf{\Lambda}\mathbf{L}$$

where

- $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ is a diagonal matrix
- $\mathbf{L}$ an arbitrary tensor.

we could pre-multiply (1a) by $\mathbf{L}$ and obtain,

$$\mathbf{L}\mathbf{q}_{,t} + (\mathbf{L}\mathbf{A})\mathbf{q}_{,x} = \mathbf{L}\mathbf{s}(\mathbf{q}, x, t) \quad \Rightarrow$$
$$(\mathbf{L}\mathbf{q})_{,t} + (\mathbf{\Lambda}\mathbf{L})\mathbf{q}_{,x} = \mathbf{s}^{\omega}(\mathbf{q}, x, t) \quad \Rightarrow$$
$$(\mathbf{L}\mathbf{q})_{,t} + (\mathbf{\Lambda})(\mathbf{L}\mathbf{q})_{,x} = \mathbf{s}^{\omega}(\mathbf{q}, x, t) \quad \Rightarrow$$

Systems of 1st order PDEs Characteristic values

$$\boxed{\omega_{,t} + (\boldsymbol{\Lambda})\omega_{,x} = \mathbf{s}^{\omega}(\mathbf{q}, x, t)} \qquad (2)$$

The expanded form of this equation is,

$$\omega_{i,t} + \lambda_i \omega_{i,x} = s_i^{\omega}(x, t), \quad \text{for } i \leq n$$

which can be independently solved for each ω_i since the equations for ω are decoupled from $\boldsymbol{\Lambda}$ being diagonal

ω are called characteristic variables:

$$\boxed{\omega = \mathbf{Lq} \quad \text{characteristic variables}} \qquad (3)$$

Systems of 1st order PDEs
Eigenvalue problem for characteristic values

- How do we find $\mathbf{L}$ and $\mathbf{\Lambda}$?
  - A left eigenvector $\mathbf{l}$, eigenvalue λ pair of the matrix $\mathbf{A}$ are defined from,

    $$\mathbf{l}\mathbf{A} = \lambda \mathbf{l}$$

    where $\mathbf{l}$ is a **row vector**.
  - **IF matrix A has n LINEARLY-INDEPENDENT eigenvectors** $\mathbf{l}^i$ (with REAL eigenvalues λ_i) we can form tensors $\mathbf{L}$ and $\mathbf{\Lambda}$ as,

$$\mathbf{L} = \begin{bmatrix} \mathbf{l}^1 \\ \vdots \\ \mathbf{l}^i \\ \vdots \\ \mathbf{l}^n \end{bmatrix} = \begin{bmatrix} \mathbf{l}^1_1 & \cdots & \mathbf{l}^1_n \\ \vdots & \ddots & \vdots \\ \mathbf{l}^i_1 & \cdots & \mathbf{l}^i_n \\ \vdots & \ddots & \vdots \\ \mathbf{l}^n_1 & \cdots & \mathbf{l}^n_n \end{bmatrix}$$

(rows of $\mathbf{L}$ are left eigenvectors of $\mathbf{A}$)

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$\mathbf{\Lambda}$ is the diagonal matrix from λ_i

Systems of 1st order PDEs
Initial conditions and solution for q

- How does the initial condition transfer from $\mathbf{q}$ to initial condition for $\boldsymbol{\omega}$?

  Since we have $\boldsymbol{\omega} = \mathbf{L}\mathbf{q}$, the ID $\mathbf{q}(x,0) = \mathbf{q}_0(x)$ becomes,

$$\boldsymbol{\omega}(x,0) = \boldsymbol{\omega}_0(x) = \mathbf{L}\mathbf{q}_0(x)$$

- Once we solve (2), that is we have $\boldsymbol{\omega}(x,t)$, how do we represent the solution in terms of primary unknown vector $\mathbf{q}$?

  Note $\boldsymbol{\omega} = \mathbf{L}\mathbf{q}$ and $\mathrm{L} \neq 0$ (why?) thus $\mathbf{q}(x,t) = \mathbf{L}^{-1}\boldsymbol{\omega}(x,t)$.

Systems of 1st order PDEs
Solution process

Procedure for solving (1),

$$\text{PDE}: \quad \mathbf{q}_{,t} + \mathbf{A}\mathbf{q}_{,x} = \mathbf{s}(\mathbf{q}, x, t)$$
$$\text{IC}: \quad \mathbf{q}(x,0) = \mathbf{q}_0(x)$$

- Step 1: Solve for n left eigenvectors $\mathbf{l}^i$ eigenvalues λ_i of $\mathbf{A}$

$$\mathbf{l}^i\mathbf{A} = \lambda^i\mathbf{l}^i \quad \text{(no summation on } i\text{)}$$

And form left-eigenvalue matrix $\mathbf{L} = [\mathbf{l}^1 \; \cdots \; \mathbf{l}^n]^{\mathrm{T}}$, $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$.

If n INDEPENDENT eigenvalues do not exist OR λ_i ARE NOT REAL (1) is not hyperbolic (need another solution method)

Systems of 1st order PDEs
Solution process

- Step 2: Define characteristic values, and corresponding source terms, and IC

Characteristic variables	$\boldsymbol{\omega} = \mathbf{Lq} \quad \Rightarrow$
Source term vector	$\mathbf{s}^{\omega}(\omega, x, t) = \mathbf{Ls}(\mathbf{L}^{-1}\boldsymbol{\omega}, x, t)$
IC	$\boldsymbol{\omega}(x, 0) = \boldsymbol{\omega}_0(x) = \mathbf{Lq}_0(x)$

Solve n-decoupled first order PDEs in ω_i

$$\left.\begin{array}{ll} \text{PDE}: & \omega_{i,t} + \lambda_i \omega_{i,x} = \mathbf{s}_i^{\omega}(\omega, x, t) \\ \text{IC}: & \omega_i(x, 0) = \omega_{i0}(x) \end{array}\right\} \quad i \leq n \quad (\text{no summation on } i) \qquad (4)$$

Note: If the source term $\mathbf{s}(\mathbf{q}, x, t)$ explicitly depends on $\mathbf{q}$ equations (4) for ω_i are coupled through the source term $\mathbf{s}^{\omega}$ but we still have n characteristics and the solution is simpler in this space.

- Step 3: Once the solution $\boldsymbol{\omega}(x, t)$ is obtained we solve for $\mathbf{q}(x, t)$ from,

$$\mathbf{q}(x, t) = \mathbf{A}^{-1}\boldsymbol{\omega}(x, t)$$

Systems of 1st order PDEs
Example: 1D elastodynamics problem

Example: Consider the solution to 1D solid mechanics problem corresponding to the conservation law $\sigma_{,x} + \rho b = p_{,t}$ where $\sigma = E\epsilon$ is stress, ρ is density, b is body force, $p = \rho v$ is linear momentum, $\epsilon = u_{,x}$ is strain, $v = u_{,t}$ is velocity, and u is displacement.

To solve this problem as a system of first order PDEs we follow these steps:

- Write equation of motion in conservation law form,

$$p_{,t} - \sigma_{,x} = \rho b \tag{5}$$

- We have two unknowns p and σ and can define

$$\mathbf{q} = \begin{bmatrix} p \\ \sigma \end{bmatrix}$$

Systems of 1st order PDEs
Example: 1D elastodynamics problem

- We need to define a second equation that is of the form $\sigma_{,t} + (?)p_{,x} = (?)$ to close the system of first order PDEs.

  Note $v_{,x} = (u_{,t})_{,x} = (u_{,x})_{,t} = \epsilon_{,t}$ so by noting that (E, ρ) are constant in this example we have:

$$\sigma_{,t} = (E\epsilon)_{,t} = E(\epsilon)_{,t} = E(v)_{,x} = E\left(\frac{p}{\rho}\right)_{,x} = \frac{E}{\rho}p_{,x} \qquad (6)$$

- We write (5) and (6) in the conservation law form $\mathbf{q}_{,t} + \mathbf{A}\mathbf{q}_{,x} = \mathbf{s}$ where

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ -\frac{E}{\rho} & 0 \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} \rho b \\ 0 \end{bmatrix}$$

Systems of 1st order PDEs
Example: 1D elastodynamics problem

- Find $\mathbf{L}$ and $\mathbf{\Lambda}$ as,

$$\mathbf{L} = \begin{bmatrix} c & 1 \\ -c & 1 \end{bmatrix}, \quad \mathbf{\Lambda} = \mathrm{diag}(-c, c) = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix}$$

  where $c = \sqrt{\frac{E}{\rho}}$ is the elastic wave speed.

- By transforming the source term and boundary conditions $\boldsymbol{\omega}_0 = \mathbf{L}\mathbf{q}_0 = \mathbf{L}[p_0(x)\ \sigma_0(x)]^{\mathrm{T}}$ we obtain,

$$\text{PDE}: \quad \omega_{1,t} - c\omega_{1,x} = c\rho b \qquad \text{IC}: \quad (\omega_1)_0(x) = cp_0(x) + \sigma_0(x) \tag{7a}$$

$$\text{PDE}: \quad \omega_{2,t} + c\omega_{2,x} = -c\rho b \qquad \text{IC}: \quad (\omega_2)_0(x) = -cp_0(x) + \sigma_0(x) \tag{7b}$$

- Finally by noting that $\mathbf{L}^{-1} = \frac{1}{2c}\begin{bmatrix} 1 & -1 \\ c & c \end{bmatrix}$ and $\mathbf{q} = [p\ q]^{\mathrm{T}} = \mathbf{A}^{-1}\boldsymbol{\omega}$ we obtain,

$$p(x) = \frac{1}{2c}\left(\omega_1(x,t) - \omega_2(x,t)\right) \tag{8a}$$

$$\sigma(x) = \frac{1}{2}\left(\omega_1(x,t) + \omega_2(x,t)\right) \tag{8b}$$

Systems of 1st order PDEs
1D elastodynamics (no body force solution)

Simple illustration when $\rho b = 0$:

- From (7) for $\rho b = 0$ we have,

$$\omega_1(x,t) = (\omega_1)_0(x+ct) \quad \omega_2(x,t) = (\omega_2)_0(x-ct) \tag{9}$$

- By using ICs from (7), (8) and above equation we get,

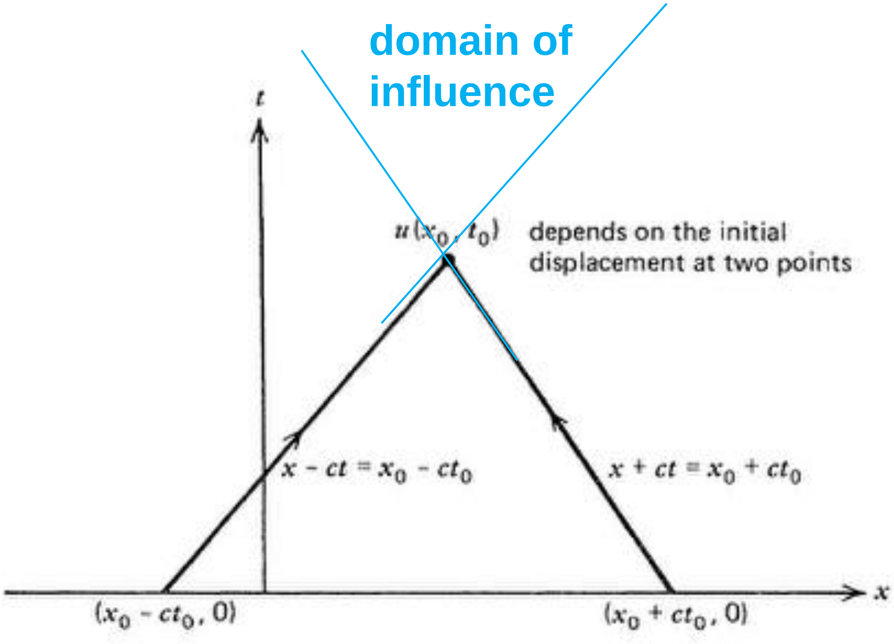
$$\begin{aligned} p(x) &= \rho v(x) = \rho u_{,t}(x) \\ &= \frac{1}{2}\left(p_0(x+ct) + p_0(x-ct)\right) + \frac{1}{2c}\left(\sigma_0(x+ct) - \sigma_0(x-ct)\right) && \text{(10a)} \\ \sigma(x) &= E\epsilon(x) = Eu_{,x}(x) \\ &= \frac{c}{2}\left(p_0(x+ct) - p_0(x-ct)\right) + \frac{1}{2}\left(\sigma_0(x+ct) + \sigma_0(x-ct)\right) && \text{(10b)} \end{aligned}$$

- This is the end of the solution to the system of first order PDEs. However, if want to obtain the displacement $u(x,t)$ we need to integrate (10b) in x. By skipping some details the integral is,

Systems of 1st order PDEs
1D elastodynamics (no body force solution)

$$u(x,t) = \frac{1}{2}\left(u_0(x-ct) + u_0(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(\xi)\mathrm{d}\xi \qquad (11)$$

This is the D'Alembert solution we obtain from $u_{,tt} - c^2 u_{,xx} = 0$ in previous slides, which is also the equation elastodynamic 1D problem discussed above: $\sigma_{,x} = p_{,t}$ $(\rho b = 0)$, $\sigma = E u_{,x}$, $p = \rho u_{,t} \Rightarrow u_{,tt} - c^2 u_{,xx} = 0$ for $c = \sqrt{\frac{E}{\rho}}$.



Systems of 1st order PDEs
Hyperbolicity condition

- Reminder: To solver the previous system of 1st order PDEs we should have been able to obtain matrix L for diagonalizing A in terms of Λ:

$$\mathbf{LA} = \mathbf{\Lambda L} \qquad \mathbf{L} = \begin{bmatrix} \mathbf{l}^1 \\ \vdots \\ \mathbf{l}^i \\ \vdots \\ \mathbf{l}^n \end{bmatrix} = \begin{bmatrix} l_1^1 & \cdots & l_n^1 \\ \vdots & \ddots & \vdots \\ l_1^i & \cdots & l_n^i \\ \vdots & \ddots & \vdots \\ l_1^n & \cdots & l_n^n \end{bmatrix} \qquad \mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

- To obtain L (diagonalizing A) we should have
  - There are n linearly independent (left) eigenvectors
  - The corresponding eigenvalues λ_i are real.
- NOTE:

$$\begin{aligned} &\mathbf{LA} = \mathbf{\Lambda L} && \mathbf{L} \text{ is left eigenvector matrix} && \Leftrightarrow \\ &\mathbf{A} = \mathbf{L}^{-1}\mathbf{\Lambda L} && \mathbf{A} \text{ is diagonalizable} && \Leftrightarrow \\ &\mathbf{AR} = \mathbf{R\Lambda} && \mathbf{R} = \mathbf{L}^{-1} \text{ is right eigenvector matrix} && \end{aligned}$$

- Hyperbolicity condition requires that we can find n characteristic values for the n-tuple q where information propagates along characteristics.

Geometric and algebraic multiplicity

- For an $n \times n$ matrix $\mathbf{A}$ we obtain eigenvalues from the n^{th} characteristic polynomial,

$$\det(A - \lambda \mathbf{I}) = 0 \qquad (14)$$

- This equation ALWAYS has n complex roots (which clearly some of all can be real). If a root λ is repeated k times we call k, the algebraic multipicity of that root.

- Given that some roots may be repeated, we list roots in ascending order,

$$\lambda_1 < \lambda_2 < \cdots < \lambda_m, \quad n^A(A) := m \leq n$$

  where m is the number of distinct roots of $\mathbf{A}$ shown by $n^A(A)$. Note that some roots may be repeated multiple times.

- The algebraic multipicity of root number k λ_k is shown by $n_k^A(A)$ which in short is shown by n_k^A.

Geometric and algebraic multiplicity

- Since (14) has n roots, even if $m < n$ (*i.e.*, some roots are repeated) we always have,

$$\sum_{k=1}^{m} n_k^A = n \qquad \text{if } m = n \quad \Rightarrow \quad n_k^A = 1 \text{ for all } k \leq n$$

- The Geometric multipicity of λ_k n_k^G is the geometric dimension (*i.e.*, number of linearly independent eigenvectors $\mathbf{u}_k^j$ of λ_k). $\mathbf{u}_k^j$ form a basis for the vector space spanned by eigenvectors of λ_k.

  Note that we can have different members in a basis of a vector space but the dimension of the vector space is independent on which basis is used.

- We have the following observations and definitions,

$n_k^G = \text{geom. multipicity } \lambda_k$	$\dim(\text{eigenspace of } \lambda_k) = \dim\{\mathbf{u} \mid \mathbf{A}\mathbf{u} = \lambda_k \mathbf{u}\}$.
$n_k^G \leq n_k^A$	Can be smaller if $n_k^A > 1$.
$n_k^G = 1 \text{ if } n_k^A = 1$	Each distinct λ has ONE eigenvector direction.
$n^G(\mathbf{A}) := \sum_{1}^{m} n_k^G$	sum of the dimensions of eigenspaces.
$n^G(\mathbf{A}) \leq n^A(\mathbf{A}) = m$	
$n^G(\mathbf{A}) = n \text{ if } n^A(\mathbf{A}) = n$	

Geometric and algebraic multiplicity
Symmetric tensors

- For symmetric (Hermitian for complex matrices) $\mathbf{A}$ we have the following properties,
  - $n^G(\mathbf{A}) = n$ ALWAYS true even if $n^A(\mathbf{A}) < n$.
  - $n^A(\mathbf{A})$ can be smaller than n.
  - ALL λ_k are REAL for $k \leq n^A(\mathbf{A})$.
  - Eigenspaces of λ_i, λ_j are mutually orthogonal and within each eigenspace an orthonormal basis can be chosen.

Geometric and algebraic multiplicity Examples

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \quad \Rightarrow (\lambda - 3)^2(\lambda - 2) = 0 \quad \Rightarrow$$

$$\lambda_1 = 2 \quad n_1^A = n_1^G = 1, \quad \text{vector space of } \lambda_1 = \text{span}[\mathrm{e}_3].$$

$$\lambda_2 = 3 \quad n_2^A = n_2^G = 2, \quad \text{vector space of } \lambda_2 = \text{span}[\mathrm{e}_1, \mathrm{e}_2].$$

$$n^A(\mathbf{A}) = 2$$

$$n^G(\mathbf{A}) = 3$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \quad \Rightarrow (\lambda - 1)^2 = 0 \quad \Rightarrow$$

$$\lambda_1 = 1 \quad n_1^A = 2, n_1^G = 1, \quad \text{vector space of } \lambda_1 = \text{span}[\mathrm{e}_1].$$

$$n^A(\mathbf{A}) = 1$$

$$n^G(\mathbf{A}) = 1 < 2 \quad \mathbf{A} \text{ is NOT diagonalizable.}$$

Hyperbolicity of a system of 1st order PDEs
Geometric and algebraic multiplicity

- Hyperbolicity of $\mathbf{q}_{,t} + \mathbf{A}(\mathbf{q}, x, t)\mathbf{q}_{,x} = \mathbf{s}(\mathbf{q}, x, t)$ for a given point $(x^\star, t^\star)$.
  - $\mathbf{A}$ is diagonalizable iff $n^G(\mathbf{A}) = n$.
  - System is hyperbolic if $\mathbf{A}$ is diagonalizable $(n^G(\mathbf{A}) = n)$ AND ALL eigenvalues are real.
  - Hyperbolicity is STRONG if $n^A(\mathbf{A}) = n$ (all characteristic values are distinct).
  - Hyperbolicity is WEAK if $n^A(\mathbf{A}) < n$ (repeated characteristic values).
  - If system is quasilinear all $\mathbf{q}$ must be considered for $\mathbf{A} = \mathbf{A}(\mathbf{q}, x^\star, t^\star)$ in definitions above.

Systems of 1st order PDEs
Hyperbolicity condition: Summary

2.9 Hyperbolicity of Linear Systems (Also for semi-linear case)

Definition 2.1. *A linear system of the form*

$$q_t + Aq_x = 0$$

is called hyperbolic *if the $m \times m$ matrix A is diagonalizable with real eigenvalues.*

We denote the eigenvalues by

$$\lambda^1 \le \lambda^2 \le \cdots \le \lambda^m$$

The matrix is diagonalizable if there is a *complete* set of eigenvectors, i.e., if there are nonzero vectors $r^1, r^2, \ldots, r^m \in \mathbb{R}^m$ such that

$$Ar^p = \lambda^p r^p \quad \text{for } p = 1, 2, \ldots, m, \tag{2.70}$$

and these vectors are linearly independent. In this case the matrix

$$R = [r^1 | r^2 | \cdots | r^m],$$

$$R^{-1}AR = \Lambda \quad \text{and} \quad A = R\Lambda R^{-1},$$

Source: [LeVeque, 2002, 2.9]

Systems of 1st order PDEs
Hyperbolicity condition: Summary

Quasi-linear system $q_t(x,t) + f(q(x,t))_x = 0.$

$$q_t + A(q,x,t)q_x = 0$$ where $$A(\mathbf{q},x,t) = \nabla_{\mathbf{q}} f(\mathbf{q},x,t)$$

is said to be *hyperbolic* at a point (q,x,t) if the matrix $A(q,x,t)$ satisfies the hyperbolicity condition (diagonalizable with real eigenvalues) at this point.

The nonlinear conservation law ( ) is *hyperbolic* if the Jacobian matrix $f'(q)$ appearing in the quasilinear form ( ) satisfies the hyperbolicity condition for each physically relevant value of q.

Source: [LeVeque, 2002, 2.11]

Systems of 1st order PDEs
More than 2 independent parameters (2D, 3D)

- Consider the system,

$$\mathbf{q}_{,t} + \mathbf{A}^1 \mathbf{q}_{x_1} + \mathbf{A}^2 \mathbf{q}_{x_2} + \mathbf{A}^3 \mathbf{q}_{x_3} = s(\mathbf{q}, \mathbf{x}, t) \tag{12}$$

where $\mathbf{x} = (x_1, x_2, x_3)$.

- In general we cannot solve this system by diagonalizing the system and solving ODEs as a system with 2 independent parameters

$$\mathbf{q}_{,t} + \mathbf{A}^1 \mathbf{q}_{x_1} = s(\mathbf{q}, x_1, t)$$

however, even in 2D & 3D if the IC, BC are 1D and the form of matrices accommodates the direction of solution implied by IC and BC we can basically solve a 1D problem.

- The hyperbolicity is investigated by seeking planar waves in direction $\mathbf{n} = (n_1, n_2, n_3)$:

$$\boxed{\mathbf{q} = \mathbf{U} f(\mathbf{n}.\mathbf{x} - ct)} \tag{13}$$

Systems of 1st order PDEs
More than 2 independent parameters (2D, 3D)

where

$\mathbf{U} = [U_1 \ U_2 \ \cdots U_n]^{\mathrm{T}}$	wave shape (mode
$\mathbf{n} = (n_1, n_2, n_3)$	wave direction
c	wave speed
f	a scalar function $(\mathbf{U})$ turns f into the vector form $\mathbf{q}$

- By plugging (13) in (12) we obtain,

$$(\mathbf{A}^n - cI)\,\mathbf{U} = 0, \quad \text{where} \quad \mathbf{A}^n := n_1\mathbf{A}^1 + n_2\mathbf{A}^2 + n_3\mathbf{A}^3$$

That is we are solving an eigenvalue problem for $\mathbf{A}^n$ exactly similar to 1D case.

- Hyperbolicity condition:

System (12) admits propagating planar waves for **arbitrary** directions $\mathbf{n}$ $\Leftrightarrow$ $\mathbf{A}^n$ is diagonalizable for **arbitrary** directions $\mathbf{n}$

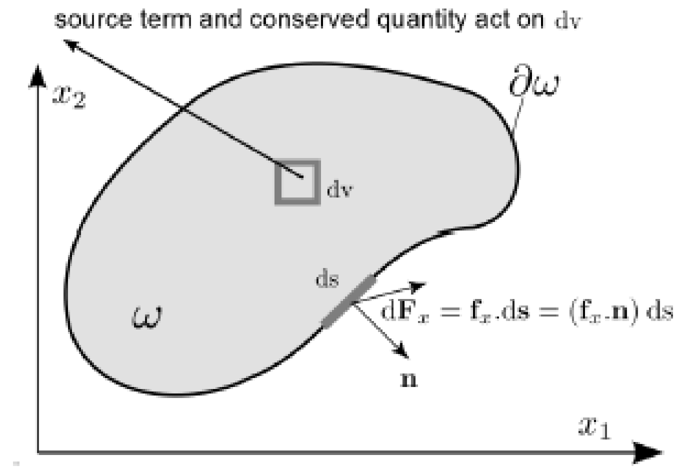
- Clearly, the same procedure works for 2D and higher dimensions as well.
- For more discussion refer to [LeVeque, 2002, section 18] (particularly 18.5).

Quasi-linear systems of 1st order PDEs
Balance laws

For a general conservation law let:

- f_t: conserved quantity = temporal flux
- f_x: total outward spatial flux
- r: source term

then the balance law for dynamics reads:



$$\forall\omega \subset \mathcal{D} \wedge \forall t : \int_\omega \mathrm{r}\,\mathrm{dv} - \int_{\partial\omega} \mathrm{f}_x.\mathrm{ds} = \int_\omega \mathrm{r}\,\mathrm{dv} - \int_{\partial\omega} (\mathrm{f}_x.\mathbf{n})\,\mathrm{ds} = \frac{\mathrm{d}}{\mathrm{dt}} \int_\omega \mathrm{f}_t\,\mathrm{dv} \qquad (13)$$

- Application of divergence theorem on the flux term yields,

$$\int_{\partial\omega} \mathrm{f}_x \mathbf{n}.\mathrm{d}s = \int_\omega \nabla.\mathrm{f}_x \mathrm{d}v$$

- By taking $\frac{\partial(.)}{\partial t}$ inside the integral in the balance law we have,

$$\forall\omega \subset \mathcal{D} \int_\omega \left\{ \frac{\partial \mathrm{f}_t}{\partial t} + \nabla.\mathrm{f}_x - r \right\} \mathrm{d}v = 0$$

Quasi-linear systems of 1st order PDEs
Strong form of balance laws

- Since $\omega \subset \mathcal{D}$ is arbitrary, by using the localization theorem we obtain the strong form of the problem,

$$\boxed{\frac{\partial \mathbf{f}_t}{\partial t} + \nabla . \mathbf{f}_x - r = 0 \quad \text{that is in 3D} \quad \mathbf{f}_{t,t} + \mathbf{f}_{1,1} + \mathbf{f}_{2,2} + \mathbf{f}_{3,3} = r}$$

  where f_1, f_2, f_3 are the components of the spatial flux $\mathbf{f}_x = [f_1, f_2, f_3]$.

- Quasilinear systems: If $\mathbf{f}_t$ or $\mathbf{f}_x$ depend on the unknown vector $\mathbf{u}$ in addition of $\mathbf{x}$ and t the system of PDE is linear. We can write it in the form,

$$\mathbf{f}_{t,t} + \mathbf{f}_{1,1} + \mathbf{f}_{2,2} + \mathbf{f}_{3,3} = r \qquad \Leftrightarrow \qquad \mathbf{A}_t\mathbf{u}_{,t} + \mathbf{A}_1\mathbf{u}_{,1} + \mathbf{A}_2\mathbf{u}_{,2} + \mathbf{A}_3\mathbf{u}_{,3} = 0 \quad \text{where}$$

$$\mathbf{A}_t(\mathbf{u}, \mathbf{x}, t) := \nabla_{\mathbf{u}} \mathbf{f}_t \quad \text{that is} \quad (\mathbf{A}_t)_{ij} = \frac{\partial (f_t)_i}{\partial u_j} \quad \text{similarly}$$

$$\mathbf{A}_i(\mathbf{u}, \mathbf{x}, t) := \nabla_{\mathbf{u}} \mathbf{f}_i \quad \text{that is} \quad (\mathbf{A}_i)_{ij} = \frac{\partial (\mathbf{f}_i)_i}{\partial u_j} \quad i \leq 3$$

  If $\mathbf{A}_t$ and $\mathbf{A}_i$ ($i \leq 3$) do not depend on $\mathbf{u}$ the system is linear or semi-linear; otherwise it is quasi-linear.

Quasi-linear systems of 1st order PDEs
Strong form of balance laws

- If further $\mathbf{u}$ is chosen to be the temporal flux $\mathbf{f}_t$ is the primary field $\mathbf{u}$ then $\mathbf{A}_t = \mathbf{I}$ and we have,

$$\boxed{\mathbf{u}_{,t} + \mathbf{f}_{1,1} + \mathbf{f}_{2,2} + \mathbf{f}_{3,3} = r \qquad \Leftrightarrow \qquad \mathbf{u}_{,t} + \mathbf{A}_1\mathbf{u}_{,1} + \mathbf{A}_2\mathbf{u}_{,2} + \mathbf{A}_3\mathbf{u}_{,3} = 0}$$

- For a scalar system clearly $\mathbf{A}_i$ are scalar.
- In 1D $\mathrm{f}_2 = \mathrm{f}_3 = 0$ $(\mathbf{A}_2 = \mathbf{A}_3 = 0)$.
- Consider a general scalar quasilinear conservation law,

$$\boxed{\begin{array}{l} u_{,t} + f_{1,1}(u) + f_{2,2}(u) + f_{3,3}(u) = r \qquad \Leftrightarrow \\ u_{,t} + g_1(u,x,t)u_{,1} + g_2(u,x,t)u_{,2} + g_3(u,x,t)u_{,3} = 0 \end{array}}$$

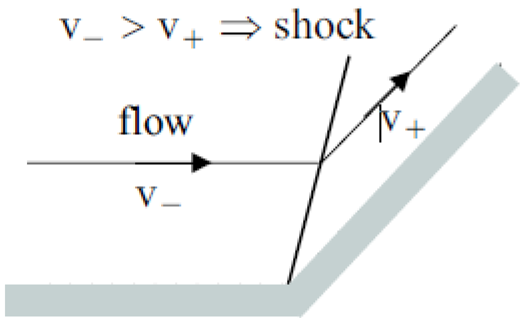
which for 1D it simply is

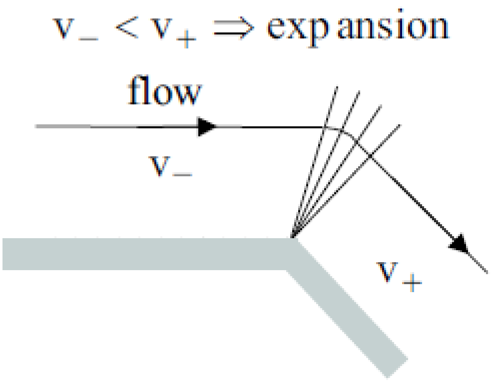
$$u_{,t} + f_{,x}(u) = r \qquad \Leftrightarrow \qquad u_{,t} + g(u,x,t)u_{,x} = 0$$

- If $g(u,x,t)$ depends on u the solution behavior for u can be very different from linear/semilinear first order systems and may exhibit shocks and expansion waves.

Quasi-linear 1st order PDEs
Formation of shocks & expansion wave

Motivation: Shock and expansion waves:





Example: Burger's equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0, \quad \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0\,.$$

$$f(u) = u^2/2 \quad \Rightarrow g(u) = f'(u) = u$$

- Speed of characteristics $= \frac{u}{1} = u$ depends on the solution u (and x, t in general) as opposed to semilinear 1st order PDEs where only depended on (x, t).

Quasi-linear 1st order PDEs
Formation of shocks & expansion wave

Example: Burger's equation (continued)

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = 0, \quad -\infty < x < \infty,\ t > 0$$

$$u(x,0) = \begin{cases} A, & x < 0 \\ B, & x \geq 0\,. \end{cases}$$

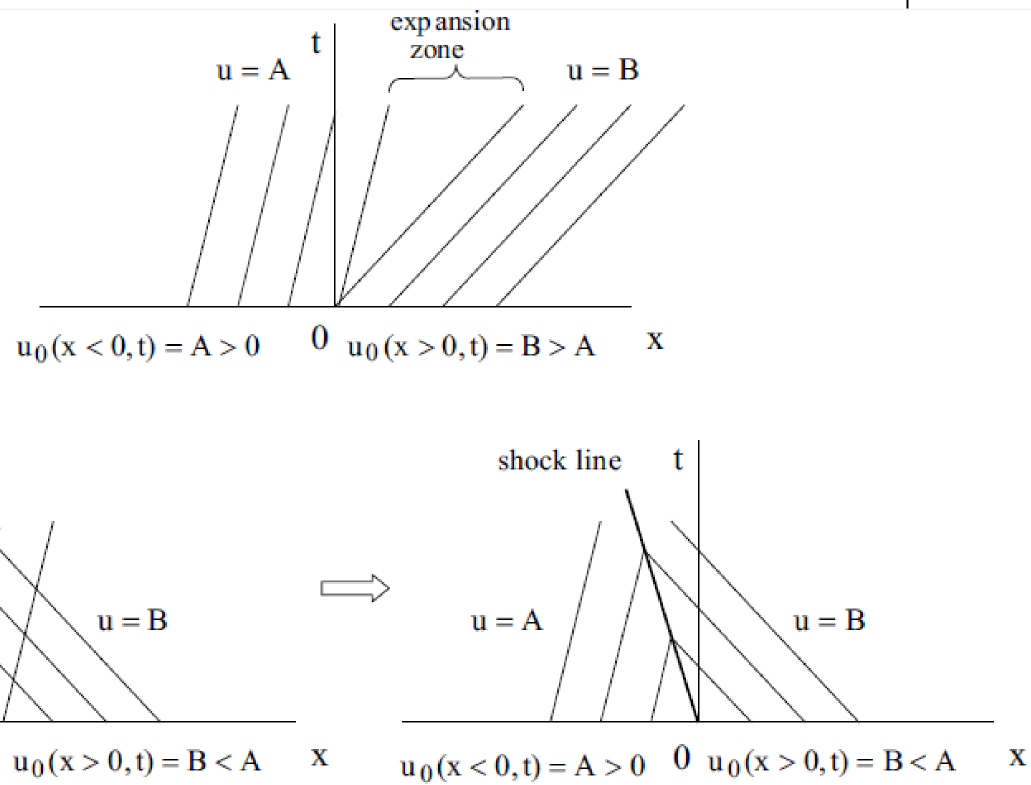


Figure IV.6 If the signal travels slower at the rear than at the front ($A < B$), the characteristic network is under-determined. Conversely, if the signal travels faster at the rear than in front ($A > B$), the characteristic network is over determined: the tentative network that displays intersecting characteristics, has to be modified to show a discontinuity line (curve).

Source: [Loret, 2008]

Quasi-linear 1st order PDEs Jump condition (brief overview)

- Quasi-linear PDE (q(u))

$$\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$$

- We define the jump operator

$$[\![\cdot]\!] = (\cdot)_+ - (\cdot)_-$$

where + and – refer to the two sides for the jump.

- If $X_s(t)$ is the position of the jump manifold in time, its equation is given by

$$\frac{dX_s(t)}{dt} = \frac{[\![q]\!]}{[\![u]\!]} = \frac{q_+ - q_-}{u_+ - u_-}$$

This is called the jump or Rankine-Hugoniot condition.

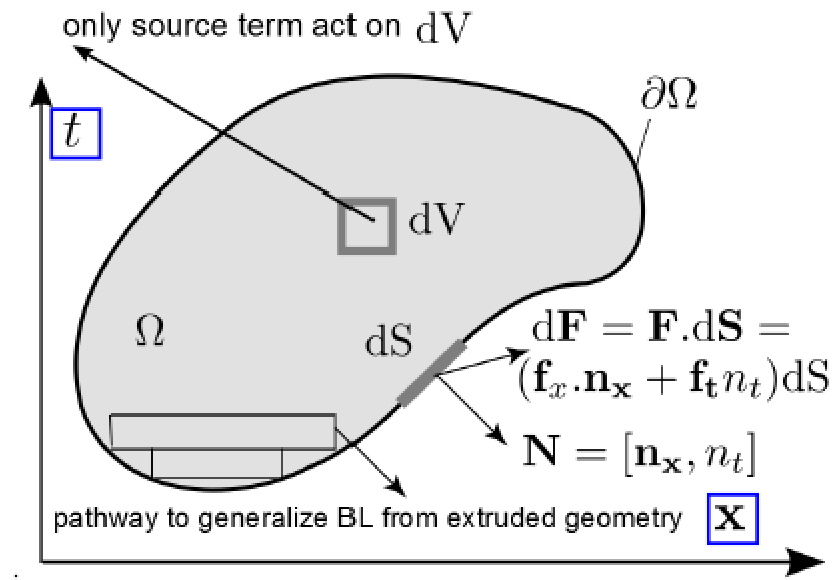
Source: [Loret, 2008]

FYI Balance laws in spacetime (graphical view)

We can define spacetime flux by combining spatial flux f_x with temporal flux f_t (e.g. $f_x = -\sigma$, $f_t = p$ in elastodynamics)

$$\mathbf{F} = [\mathbf{f}_x | \mathbf{f}_t]$$

then the balance law for dynamics reads:



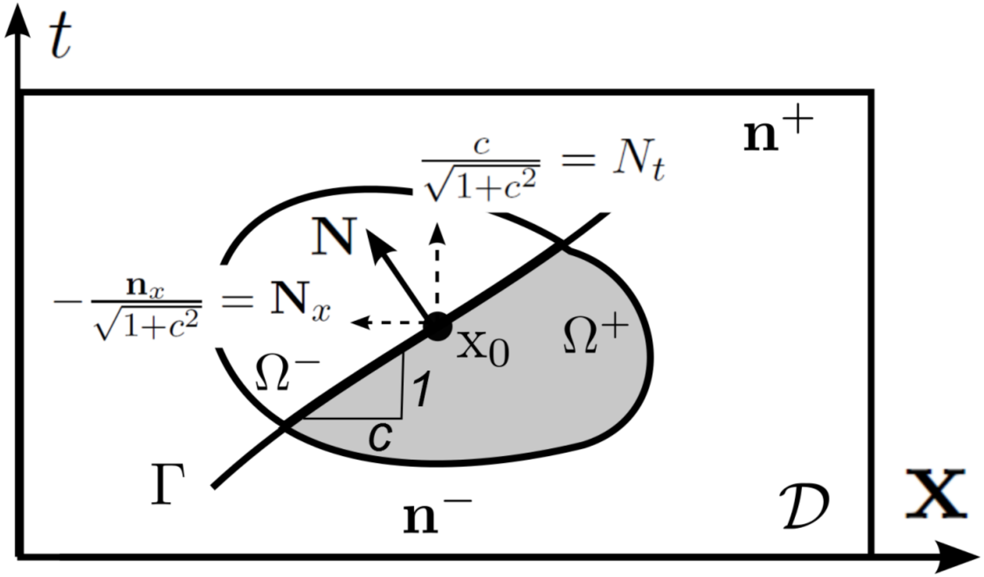
The balance law is

$$\forall \Omega \subset \mathcal{D} : \int_{\partial\Omega} \mathbf{F}.\mathrm{dS} - \int_{\Omega} \mathrm{r}\,\mathrm{dV} = \int_{\partial\Omega} (\mathbf{f}_x.\mathbf{n_x} + \mathbf{f_t} n_t)\mathrm{dS} - \int_{\Omega} \mathrm{r}\,\mathrm{dV} = 0$$

FYI Jump condition (graphical view)

By writing two balance law expressions for Ω^+ and Ω^- we obtain the jump condition

$$[\![\mathbf{F}]\!].\mathbf{N} = 0 \quad \Rightarrow$$



$$[\![\mathbf{F}]\!].\mathbf{N} = \begin{bmatrix} [\![\mathbf{f}_x]\!] \\ [\![\mathbf{f}_t]\!] \end{bmatrix} . \begin{bmatrix} \mathbf{N}_x \\ N_t \end{bmatrix} = [\![\mathbf{f}_x]\!].\mathbf{N}_x + [\![\mathbf{f}_t]\!] N_t$$

$$= [\![\mathbf{f}_x]\!].\frac{-\mathbf{n}_x}{\sqrt{1+c^2}} + [\![\mathbf{f}_t]\!]\frac{c}{\sqrt{1+c^2}} = 0 \quad \Rightarrow \quad -[\![\mathbf{f}_x]\!].\mathbf{n}_x + c[\![\mathbf{f}_t]\!] = 0 \quad \Rightarrow$$

- n_t is the spatial component of jump manifold, by 90˚ rotation from jump to normal direction we get the – sign.
- We cannot define normal vectors in spacetime, but this sketch provides an idea how the jump condition is derived

$$c = \frac{[\![\mathbf{f}_x]\!].\mathbf{n}_x}{[\![\mathbf{f}_t]\!]}$$

Rankine-Hugoniot Jump conditions

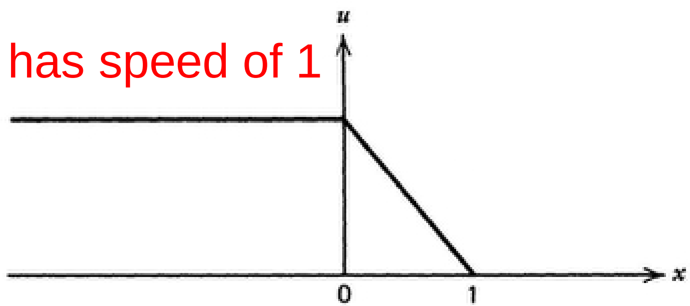
Quasi-linear 1st order PDEs
Jump formation example

Shock formation: example Traffic flow (u is speed)

$$\text{PDE} \qquad u_t + 2uu_x = 0 \qquad -\infty < x < \infty \qquad 0 < t < \infty$$

$$\text{IC} \qquad u(x,0) = \begin{cases} 1 & x \leq 0 \\ 1 - x & 0 < x < 1 \\ 0 & 1 \leq x \end{cases} \qquad -\infty < x < \infty$$

Wave has speed of 1



Initial condition

Wave has speed of 0

Source: [Farlow, 2012]

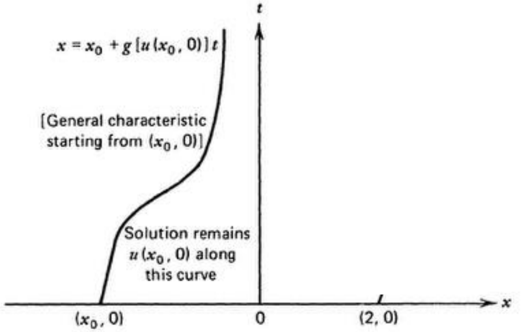
Quasi-linear 1st order PDEs
Jump formation example

For detailed solution derivation refer to [Farlow, 2012, lesson 28]

Solution:

1. Characteristics carrying u = 1 for for $x_0 \le 0$.

$$x = x_0 + g[u(x_0,0)]t = x_0 + g[1]t = x_0 + 2t \Rightarrow t = \frac{1}{2}(x - x_0)$$



2. Characteristics carrying $0 \le u \le 1$ for $0 < x_0 \le 1$

$$x = x_0 + g[u(x_0,0)]t = x_0 + g[0]t = x_0 \Rightarrow t = \frac{x - x_0}{2(1 - x_0)}$$

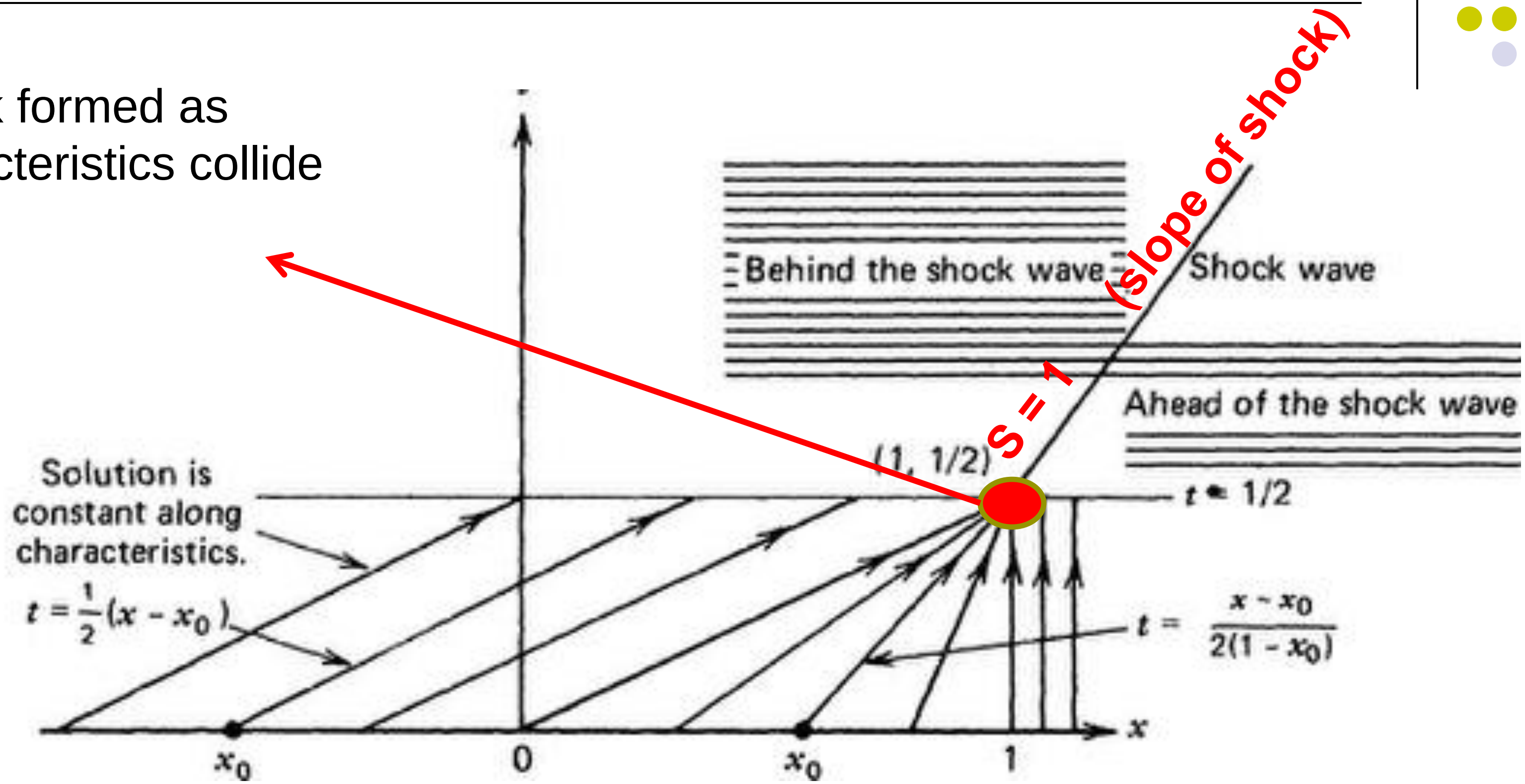
3. Characteristics carrying u = 0 for $1 < x_0$

$$x = x_0 + g[u(x_0,0)]t = x_0 + g[0]t = x_0 \Rightarrow x = x_0$$

Source: [Farlow, 2012, lesson 28]

Quasi-linear 1st order PDEs
Jump formation example

Shock formed as characteristics collide



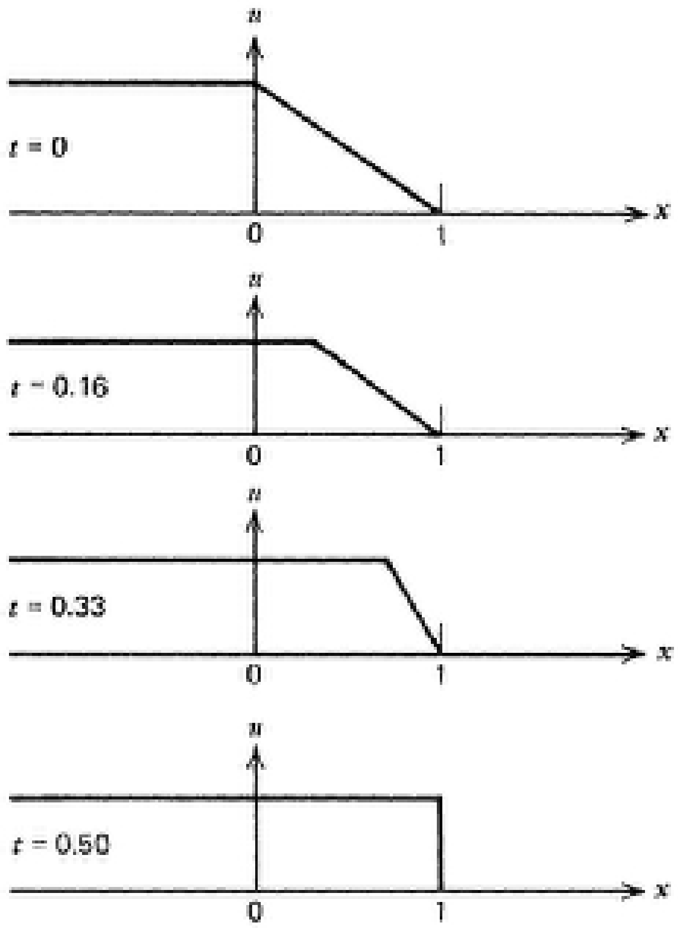
$$S = \frac{f(u_R) - f(u_L)}{u_R - u_L} = \frac{0 - 1}{0 - 1} = 1$$

Flux = u^2

Source: [Farlow, 2012, lesson 28]

Quasi-linear 1st order PDEs
Jump formation example

Stages of solution



Source: [Farlow, 2012, lesson 28]

1D Acoustic equation

Acoustic equation

- 1D acoustic equation for fluid / solid looks very much like the elastodynamic problem, especially, when the background velocity is zero. The primary field is given by,

$$\mathbf{q} = \begin{bmatrix} p \\ u \end{bmatrix} \quad (15)$$

  where

  - p: pressure ($= -\sigma$)
  - u: velocity (note in accordance with fluid mechanics convention u is used for velocity NOT displacement as in solid mechanics).

1D Acoustic equation

- Closing the system of conservation laws:
  - Compatibility condition requires $\dot{p} = -\dot{\sigma} = -Ku, x \Rightarrow \dot{q}_1 + Kq_{2,x} = 0$ where K is the bulk modulus (similar to elastic modulus in solid mechanics).
  - Balance of linear momentum requires $(\rho u)_{,t} - \sigma_{,x} = \rho\dot{u} + p_{,x} = 0 \Rightarrow \dot{q}_2 + \frac{1}{\rho}q_{1,x} = 0$.

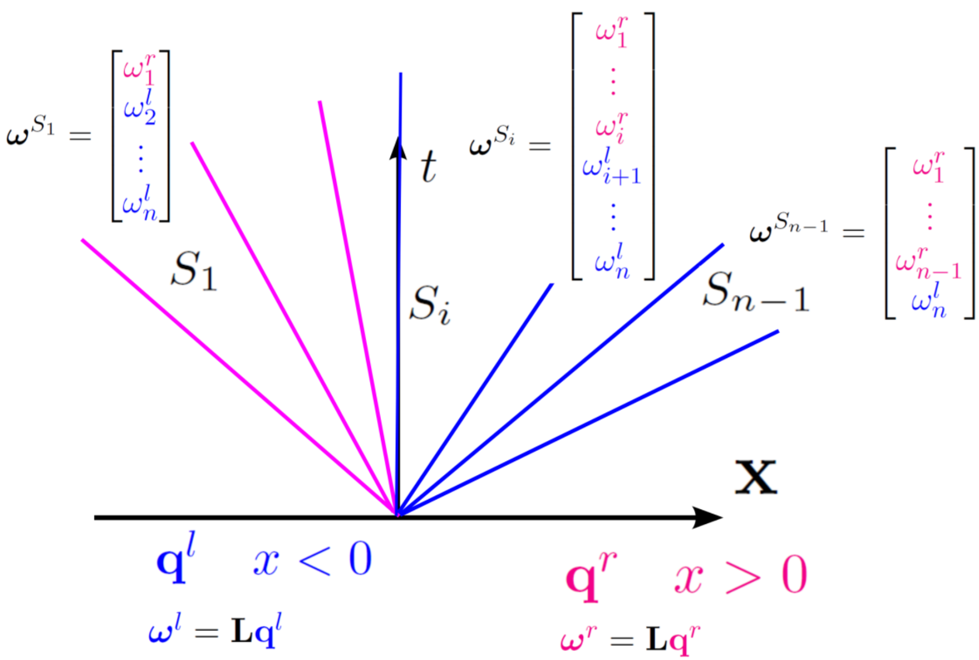
Thus the system of first order PDEs is,

$$\dot{\mathrm{q}} + \mathbf{A}\mathbf{q}_{,x} = 0, \text{ where } \mathbf{A} = \begin{bmatrix} 0 & K \\ \frac{1}{\rho} & 0 \end{bmatrix} \quad \text{(spatial flux matrix) and, } \mathbf{q} = \begin{bmatrix} p \\ u \end{bmatrix} \quad (16)$$

Riemann solution techniques
Linear conservation laws

Riemann solution for linear conservation laws:
Approach 1: Using characteristic values

Characteristic values $\omega = \mathbf{L}\mathbf{q}$ are constant (or solved as ODEs) along characteristic directions



Riemann solution techniques
Linear conservation laws

- Primary field $\mathbf{q}$ has n components

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

- Transfer to characteristic variables and directions For the system on $\mathbf{q}$ we transfer to characteristics by,

$$\text{For } \dot{\mathbf{q}} + \mathbf{A}\mathbf{q}_{,x} = 0 \quad \mathbf{L}\mathbf{A} = \mathbf{\Lambda}\mathbf{L}, \quad \text{for } \Lambda = \text{diag}(c_1, \cdots, c_n)$$

  - Characteristic variables are $\boldsymbol{\omega} = \mathbf{A}\mathbf{q}$, that is $\omega_i = A_{ij}q_j$.
  - Eigenvalues (wave speeds) are $c_1 \leq c_2 \leq \cdots \leq c_n$

- Initial conditions are

$$\mathbf{q}(x,0) = \mathbf{q}_0 = \begin{cases} \mathbf{q}^l & x < 0 \\ \mathbf{q}^r & x > 0 \end{cases} \quad \Rightarrow \boldsymbol{\omega}(x,0) = \boldsymbol{\omega}_0 = \begin{cases} \boldsymbol{\omega}^l = \mathbf{L}\mathbf{q}^l & x < 0 \\ \boldsymbol{\omega}^r = \mathbf{L}\mathbf{q}^r & x < 0 \end{cases}$$

Riemann solution techniques
Linear conservation laws

- Characteristic values $\boldsymbol{\omega} = \mathbf{Lq}$ are constant along characteristic directions. That is, ω_i is constant along the wave moving with speed c_i.

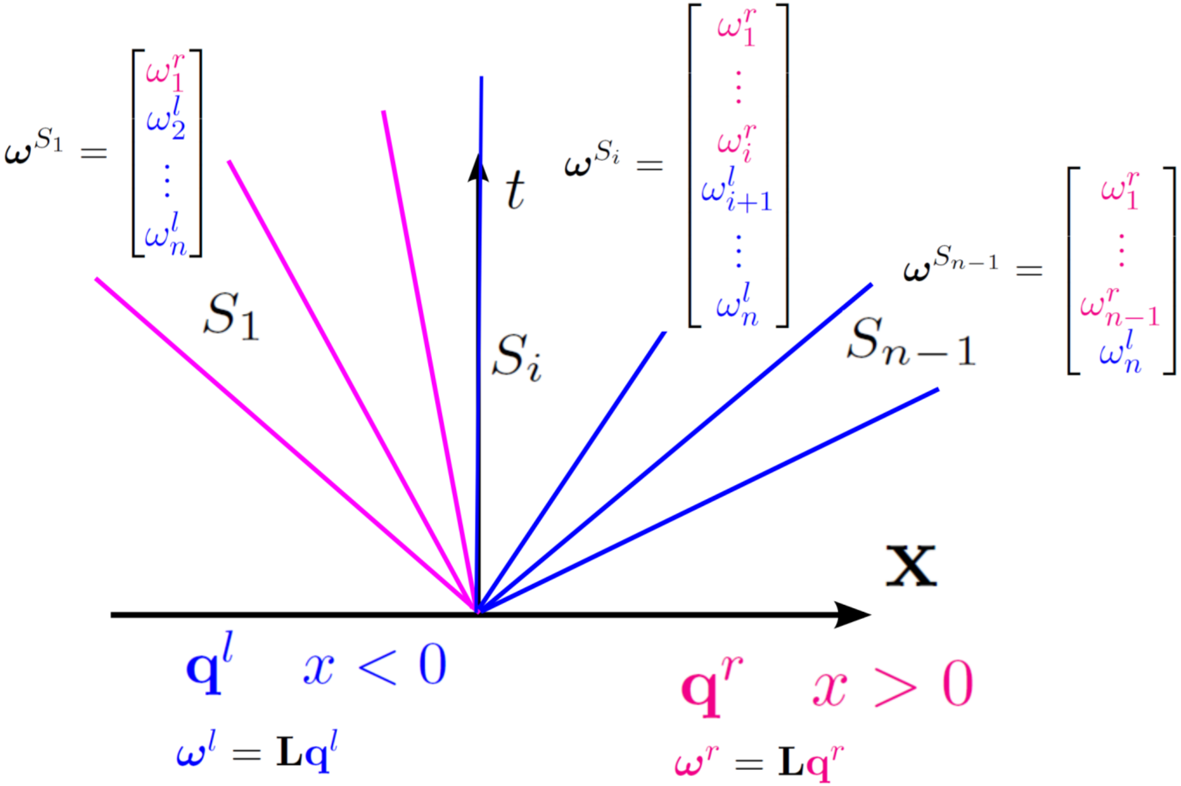
- Thus, for sample segments S^1, S^i, S^{n-1} we have,

$$\boldsymbol{\omega}^{S_1} = \begin{bmatrix} \omega_1^r \\ \omega_2^l \\ \vdots \\ \omega_n^l \end{bmatrix} \quad \boldsymbol{\omega}^{S_i} = \begin{bmatrix} \omega_1^r \\ \vdots \\ \omega_i^r \\ \omega_{i+1}^l \\ \vdots \\ \omega_n^l \end{bmatrix} \quad \boldsymbol{\omega}^{S_{n-1}} = \begin{bmatrix} \omega_1^r \\ \vdots \\ \omega_{n-1}^r \\ \omega_n^l \end{bmatrix}$$

- Transfer back to primary variables and fluxes is by using $\mathbf{L}$,

$$\mathbf{q}^{S_i} = \mathbf{L}^{-1}\boldsymbol{\omega}^{S_i}$$

Riemann solution techniques
Linear conservation laws



Riemann solution techniques
Linear conservation laws: Acoustic equation

Riemann solutions approach 1: Acoustic equation

- We obtain the characteristic values for the acoustic problem (16) by forming the left eigenvalue eigenvector pairs,

$$\mathbf{A} = \begin{bmatrix} 0 & K \\ \frac{1}{\rho} & 0 \end{bmatrix}, \mathbf{LA} = \mathbf{\Lambda L}, \quad \text{where} \quad \mathbf{L} = \begin{bmatrix} -1 & Z \\ 1 & Z \end{bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} -c & 0 \\ 0 & c \end{bmatrix} \tag{17}$$

where

$$Z = \sqrt{K\rho} = c\rho \qquad \text{Impedance} \tag{18a}$$

$$c = \sqrt{\frac{K}{\rho}} \qquad \text{Wave speed} \tag{18b}$$

- Characteristic values $\omega = \mathbf{Lq}$ are defined as,

$$\omega = \mathbf{Lq} \qquad \Rightarrow \qquad \begin{cases} \omega_1 = -p + Zu \\ \omega_2 = p + Zu \end{cases} \tag{19}$$

Riemann solution techniques
Linear conservation laws: Acoustic equation

- Direction of characteristics

$$\begin{cases} \text{Along } -c \quad \omega_1 = -p + Zu \quad \text{is constant (or varied if having source term)} \\ \text{Along } c \quad \omega_2 = p + Zu \quad \text{is constant (or varied if having source term)} \end{cases}$$

- Solution in terms of characteristics: According to the direction of characteristics the solution for the three regions shown is given by,

$$\boldsymbol{\omega}^L := \begin{bmatrix} \omega_1^l \\ \omega_2^l \end{bmatrix} \quad \Rightarrow \quad \mathbf{q}^L = \mathbf{L}^{-1}\boldsymbol{\omega}^L = \mathbf{q}^l$$

$$\boldsymbol{\omega}^R := \begin{bmatrix} \omega_1^r \\ \omega_2^r \end{bmatrix} \quad \Rightarrow \quad \mathbf{q}^R = \mathbf{L}^{-1}\boldsymbol{\omega}^R = \mathbf{q}^r$$

$$\boldsymbol{\omega}^m := \begin{bmatrix} \omega_1^r \\ \omega_2^l \end{bmatrix} = \begin{bmatrix} -p^r + Zu^r \\ p^l + Zu^l \end{bmatrix} \quad \Rightarrow \quad \mathbf{q}^m = = \mathbf{L}^{-1}\boldsymbol{\omega}^m \quad \text{that is}$$

$$\begin{bmatrix} p^m \\ u^m \end{bmatrix} = \begin{bmatrix} \frac{p^r+p^l}{2} - \frac{Z}{2}(u^r - u^l) \\ -\frac{1}{2Z}(p^r - p^l) + \frac{u^r+u^l}{2} \end{bmatrix}$$

Riemann solution techniques
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