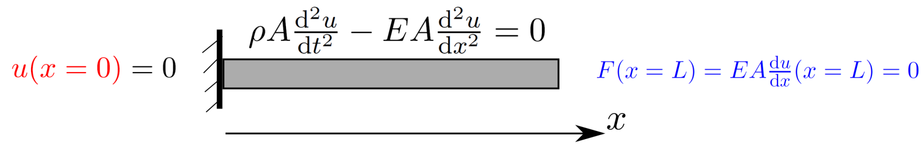
1. (**30 Points**) Consider 1D bar problem for constant area A, Young's modulus E, and density ρ,

$$\rho A \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - EA \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0 \tag{1}$$

which as shown in the figure has prescribed displacement boundary on the left and free stress on the right. This boundary condition configuration is denoted by P1F1 (1 free and 1 prescribed BCs),



(a) Show that exact natural frequencies are,

$$\omega_n = (n - \frac{\pi}{2})\frac{c}{L}, \qquad \text{where } c = \sqrt{\frac{E}{\rho}} \tag{2}$$

(b) Obtain mode shapes $\Phi_i(x)$.

Hint: Use separation of variables $u(x,t) = \Phi(x)T(t)$ and follow the same process used in the course notes for a double fixed bar example.

2. ((20 + 10 + 20 =) **50 Points**) Recalling that element stiffness and mass matrices are (refer to the course notes),

$$\mathbf{k}^e = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{3a}$$

$$\mathbf{m}_c^e = \frac{AL_e\rho}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{m}_d^e = \frac{AL_e\rho}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3b}$$

where subscripts c and d refer to consistent and diagonal (lumped) mass matrices, and L_e is the element length.

(a) Find natural modes and frequencies for one element for **both** consistent and diagonal mass matrix options.

(b) Describe why one of the natural frequencies is zero. Use corresponding natural mode to explain what the natural mode corresponds to.

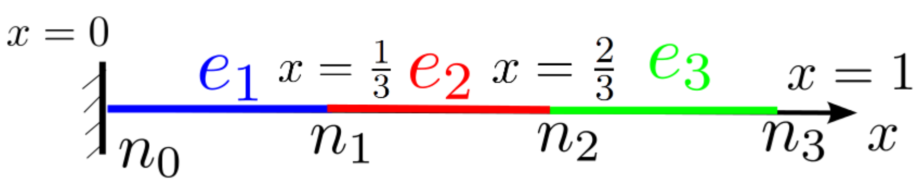
(c) Exact natural frequencies of the same element and corresponding modes will be $\omega_n = \frac{n\pi c}{L_e}$, $\Phi_n = \cos(n\pi\frac{x}{L_e})$ (you do not to prove this, the proof is similar to that of problem 1). Compare the first nonzero natural frequency that you obtain from either consistent or diagonal mass matrix options. We call them $\omega_{c\,1}^h$ for consistent mass matrix option and $\omega_{d\,1}^h$ for diagonal (lumped) mass matrix option. Compute $\omega_{c\,1}^h/\omega_1$ and $\omega_{d\,1}^h/\omega_1$ and comment on the values you obtain.. That is, discuss what the error of one element natural frequencies are with respect to the exact values.

Hint: Remember that natural frequencies for a FEM mesh with no damping ($\mathbf{C} = 0$) are obtained by solving a generalized eigenvalue problem for,

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = 0 \tag{4}$$

For one element $\mathbf{M} = \mathbf{m}_c^e$ (consistent mass matrix) or $\mathbf{M} = \mathbf{m}_d^e$ (diagonal mass matrix) and $\mathbf{K}$ is also taken from element value. Solve the corresponding natural mode, frequency problem for one element.

3. ((6 × 15 =) **90 Points**) First, the solution of a sample natural mode analysis is outlined for the one side prescribed one side free displacement boundary condition. Consider that this problem is discretized with three elements of equal size meaning that the element size is $L_e = L/3$. Also for simplicity assume $L = 1, A = 1, E = 1, \rho = 1$. Then local element matrices are (only consistent mass matrix option is shown):



$$\mathbf{m}_c^{e_1} = \frac{AL_e\rho}{6} \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array} = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} \frac{1}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{1}{9} \end{bmatrix} \end{array}, \quad \text{similarly} \quad \mathbf{m}_c^{e_2} = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} \frac{1}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{1}{9} \end{bmatrix} \end{array}, \quad \mathbf{m}_c^{e_3} = \begin{array}{cc} & \begin{array}{cc} 2 & 3 \end{array} \\ \begin{array}{c} 2 \\ 3 \end{array} & \begin{bmatrix} \frac{1}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{1}{9} \end{bmatrix} \end{array} \tag{5}$$

after assembling free degrees of freedom into the global matrix we obtain,

$$\mathbf{M}_c = \begin{bmatrix} \frac{1}{9} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{9} & \frac{1}{18} & 0 \\ \frac{1}{18} & \frac{1}{9} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{9} & \frac{1}{18} \\ 0 & \frac{1}{18} & \frac{1}{9} \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \tag{6}$$

Similarly, given that for three elements,

$$\mathbf{k}^e = \frac{AE}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

which after assembly to the global system (similar to the mass matrix) we get,

$$\mathbf{K} = 3 \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{7}$$

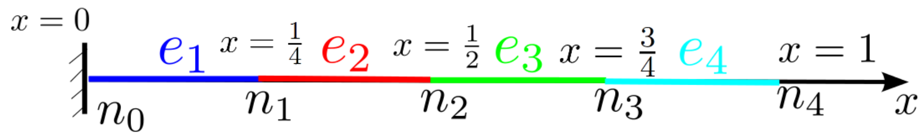
To obtain the 3 natural frequencies and modes for this 3 dof FEM model we solve the following generalized eigenvalue problem,

$$\mathbf{K}\boldsymbol{\Phi}_i^h = {\omega_i^h}^2 \mathbf{M}\boldsymbol{\Phi}_i^h, i \leq 3, \quad \text{that is} \quad \begin{bmatrix} 6 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix} = {\omega_i^h}^2 \begin{bmatrix} \frac{4}{18} & \frac{1}{18} & 0 \\ \frac{1}{18} & \frac{4}{18} & \frac{1}{18} \\ 0 & \frac{1}{18} & \frac{2}{18} \end{bmatrix} \boldsymbol{\Phi}_i^h, i \leq 3 \tag{8}$$

just for reference we get $\omega_1^h = 1.5888$ and $\omega_3^h = 9.4266$. The exact natural frequencies 1 and 3 are given from (2) ($L = c = 1$), $\omega_1 = \pi/2 = 1.5708$ and $\omega_3 = 5\pi/2 = 7.8540$. We observe that ω_1^h has much smaller error than ω_3^h.

Another quantity of great practical important is finding how the highest frequency of this 3 dof system, *i.e.*, $\omega_3^h = 9.4266$, compares with the highest frequency of its SMALLEST element (here all elements are of the same size). We call the latter $\omega_{h_{\min}}$. From the solution of problem 2 and using the correct element size (1/3) you will realize that $\omega^c_{h_{\min}} = 10.3923$ (consistent mass option). We observe, $\omega_3^h/\omega^c_{h_{\min}} < 1$.

Based on the following background answer the following questions for the four dof system shown below,



(a) Obtain 4×4 mass matrices $\mathbf{M}_c$ (consistent) $\mathbf{M}_d$ (diagonal) and stiffness matrix $\mathbf{K}$.

(b) For EACH option (consistent and diagonal mass), find all 4 natural frequencies and natural modes using the generalized eigenvalue problem. Matlab and many other packages support the solution of a generalized eigenvalue problem.

(c) List the highest natural frequency of the 4 dof system for consistent and diagonal mass matrix options. That is, $\omega^h_{c\,4}$ and $\omega^h_{d\,4}$ (These are computed in previous item). Compare them with the fourth exact natural frequency ω_4 obtain from (2). That is, provide values for $\omega^h_{c\,4}/\omega_4$ and $\omega^h_{d\,4}/\omega_4$. This is to demonstrate how accurate the last modes are.

(d) Compare the minimum element size ($L_e = 0.25$) maximum frequency for both consistent mass and diagonal mass option: $\omega^c_{h_{\min}}$, $\omega^d_{h_{\min}}$.

(e) Compute the ratio of the system's maximum computed frequency to the smallest element's maximum frequency. That is, $\omega^h_{c\,4}/\omega^c_{h_{\min}}$ and $\omega^h_{d\,4}/\omega^d_{h_{\min}}$. Comment on their values.

(f) Refer to the two files "Ratio of maximum frequency to that of the smallest element_consistent.png" and "Ratio of maximum frequency to that of the smallest element_lumped" where $\omega^h_{c\,n}/\omega^c_{h_{\min}}$ and $\omega^h_{d\,n}/\omega^d_{h_{\min}}$ are computed for an N segment bar for three different boundary conditions:

- P0F2: Two end points are free displacement condition.
- P1F1: One prescribed and one free displacement conditions (problem considred in this HW).
- P2F0: Two end points are prescribed displacement condition.

The values to the left in the horizontal axis (closes to zero) the smaller the elements are. Based on these plots and the figures provided comment on $\omega^h_{c\,n}/\omega^c_{h_{\min}}$ and $\omega^h_{d\,n}/\omega^d_{h_{\min}}$ and how these values change in a uniform FE mesh this ratio changes as elements get smaller (use the figure).

4. ((10 + 20 + 10 =) **40 Points**) In the provided files starting with "Natural frequency convergence" convergence rates of natural frequencies of mode 1, 2, and 8 and three different boundary conditions (P0F2, P1F1, P2F0), and consistent or diagonal mass matrix options are provided. The plots are generated by solving the solutions with different element sizes h so that we can numerically investigate the convergence rate of natural frequencies. Answer the following questions,

(a) From these plots discuss what is the convergence rate of ω_i^h (based on results for $\omega_1^h, \omega_2^h, \omega_8^h$) for different mass matrix options and boundary conditions. Refer to the plots for answering this question. (note: some or all convergence rates may be identical).

(b) Refer to *a priori* error estimate provided in course notes

$$0 \le \omega_i^h - \omega_i \le C h^{2(p+1-m)} \omega_i^{\frac{2p+2-m}{m}} \tag{9}$$

Note that the proof of this condition is based on having Galerkin FEM formulation (*i.e.,* consistent mass) and full integration order.

Based on the discussion in the course notes, provide values for p and m.

(c) Having p and m, from (9) discuss what the convergence rate for ω_i^h for the bar problem with linear elements should be. How does this value compare with convergence rates you obtained for modes $i = 1, 2, 8$?

5. ($(80 + 10 =)$ **90 Points**) Description of spectra plot for natural frequencies. **Part one requires computational code writing. Even without generating the results you can answer the second question of this problem with the provided plot**

This plot shows ω^h/ω (vertical axes) versus normalized wave number $\eta = n/N$.

- n is the mode number.
- N is the number of dof of the discrete FEM mesh.

For example for N dof discretization the mode one is shown in x coordinate $\eta = 1/N$ and its y value is ω_1^h/ω_1 that is the ratio of the numerical first natural frequency to the first exact natural frequency. Similarly, for the very last numerical natural frequency that this N dof FEM grid can capture, we have $n = N$. For $n = N$ x value is $\eta = N/N = 1$, and y value is ω_N^h/ω_N. For an N dof discretization of the problem we get N points $(\eta = n/N, \omega_n^h/\omega_n)$, $n = 1, \cdots, N$. A sample plot that superimposes results from different FEM dofs $N = 2, 8, 32, 128, 256$ and two different mass matrix options (consistent and diagonal) is "Spectra3_2 fixed ends.png".

(a) Use a computational tool, *e.g.,* Matlab, to generate spectral plots for P1F1 BC with results for $N = 2, 8, 32, 128$ and two mass matrix options (consistent and lumped) all super-imposed on in plot (similar to "Spectra3_2 fixed ends.png").

(b) Why $\omega^h/\omega \le 1$ when using diagonal mass matrix albeit $0 \le \omega_n^h - \omega_n$ in (9)? Explain the source of apparent discrepancy. You can use "Spectra3_2 fixed ends.png" is not solving the previous part.

6. **(EXTRA CREDIT)**($(25 + 25 =)$ **50 Points**) Explanation of the meaning of spectral plot obtained in previous question.

(a) In your own word, explain why ω^h/ω becomes only a function of $\eta = n/N$ that is **the relative accuracy of natural frequency ω^h/ω is a direct function of the relative order of natural mode the FEM can capture** $\eta = n/N$.

Hint: Consider that $\eta = 1/N$ is the lowest (and most accurate) mode an N dof system can capture, while $\eta = n/N$ for $n = N$ is the very last mode (and the least accurate) it would capture. Points that would facilitate the discussion is that for N dof system element size scales as $h = L/N$ (in 1D). At the same time, for a mode n the length scale in which the solution oscillates in space is L/n (can be easily seen in all mode n shape functions we obtained). The ratio of these two ratio is related to how accurately an N dof FEM can capture mode n?

(b) Mathematically demonstrate this point. Use the convergence equation for natural frequency (9) (written for mode n)

$$0 \le \omega_n^h - \omega_n \le C h^{2(p+1-m)} \omega_n^{\frac{2p+2-m}{m}} \tag{10}$$

- For the bar problem we considered plug in the value of m. Use this equation to demonstrate that at least for small h where the asymptotic expression (10) holds we can demonstrate ω_n^h/ω_n is a function of η only.

Hint: First divide the equation by ω_n so to get relative value ω_n^h/ω_n on the LHS. For a 1D problem of length L and uniform element size h we have $N_e h = L$ where N_e is the number of elements. Given that the number of dof N is proportional to number of elements (why?) $h \propto \frac{1}{N}$. At the same time, $\omega_n \propto n$ (at least for larger n). By these two substitutions on the RHS ($h \propto \frac{1}{N}$ and $\omega_n \propto n$) we can express RHS only in terms of n and N. Verify the simplified expression results in ω_n^h/ω_n is a function of $\eta = n/N$ only for small h (large N).