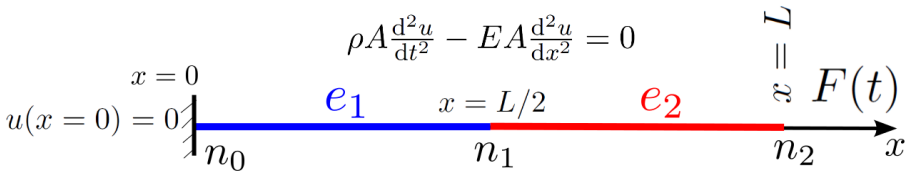
1. ($(25(a) + 4 \times 15(b) + 3 \times 5(c) + 10(d) =)$ **110 Points**) Consider 1D bar problem shown in the figure for constant area $A = 1$, Young's modulus $E = 1$, and density $\rho = 1$, and length $L = 1$. The spatial domain is discretized with two elements e_1 and e_2,



   (a) Show that stiffness matrix matrix $\mathbf{K}$, mass matrices (consistent $\mathbf{M}_c$ and diagonal (lumped) $\mathbf{M}_d$), and force vector $\mathbf{R}$ are,

$$\mathbf{K} = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}, \qquad \mathbf{M}_c = \begin{bmatrix} \frac{1}{3} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{6} \end{bmatrix}, \qquad \mathbf{M}_d = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} 0 \\ F(t) \end{bmatrix} \tag{1}$$

   You can compute local element stiffness and mass matrices (consistent and lumped) from equations (158), (159), (167) in §2.3.9 . The 1D example from §2.3.11 is a good reference on how the local matrices are assembled to the global system.

   (b) Recall that $\max_l(\lambda_l^h)$ denotes the maximum frequency of the global system (a 2 dof system herein) and λ_e^m denotes the maximum element-wise frequency (*i.e.,* for each element compute the maximum frequency and compute the maximum of those). λ_e^m is much easier to compute in practice because as described in the course notes we know what these values will be for given element types. Both $\max_l(\lambda_l^h)$ and λ_e^m can be directly computed by modal analysis of the global system and one element (worst element, *i.e.,* element with highest frequency), respectively.

   The mass matrix (either in the global or element local level) can be consistent or diagonal (lumped) mass matrix and are decorated by c and d below.

   Compute the following,

   i. $\max_l(\lambda_{c_l}^h)$: Maximum global system frequency with consistent mass matrix.
   ii. $\lambda_{c_e}^m$: Maximum element frequency with consistent mass matrix.
   iii. $\max_l(\lambda_{d_l}^h)$: Maximum global system frequency with diagonal (lumped) mass matrix.
   iv. $\lambda_{d_e}^m$: Maximum element frequency with diagonal (lumped) mass matrix.

   for element level quantities do not use the formulas in the course note and solve the values by modal analysis of the smallest element with free end points.

   (c) Answer the following questions,

   i. Is the worst element maximum frequency larger or the system maximum frequency? Use both consistent and diagonal mass matrix options.
   ii. Which mass option (consistent versus diagonal) provides a higher frequency? Use global and element level values for your comparison.
   iii. For the previous question for the element level values compare $\lambda_{d_e}^m$ and $\lambda_{d_e}^m$. Then compare them with analytical value for the maximum frequency the element can model and state which one overestimates the frequency and which one underestimate. The same assertion can be made for the global system but is not asked to be checked in this problem.

   Hint: (for Q3): For a bar of free ends (which is similar to one element boundary conditions when its frequencies are computed) analytical natural frequencies are $\omega_n = n\pi c/L_e$ where L_e is the element length and $c = \sqrt{E/\rho}$ the wave speed.

(d) Compute the system (global) Rayleigh damping matrix $\mathbf{C}$ (using consistent mass matrix $\mathbf{M}$) for $a_0 = 0.05$, $a_1 = 0.01$. You **will not add** this damping matrix to FEM equations in the next questions. For Rayleigh damping matrix refer to (213.a) which is the same as 156.

2. $(([5(i) + 5(ii) + 30(iii)](a) + [6 \times 5(i, ii, iii, iv, vi, vii) + 10(v)](b) + [105(i, iii) + 5(ii)](c) =)$ **190 Points**) Time marching of the 2 dof problem in the figure.

Solve the MDOF ODE corresponding to the problem from previous questions:

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$$

that is the system with no damping. The mass matrix $\mathbf{M}$ takes either the consistent form $\mathbf{M}_c$ or diagonal (lumped) form $\mathbf{M}_d$ depending on the method used. First part of this problem is about using the Newmark method as an example of a single-step method and the second part the central difference method as an example of a linear multi step (LMS) method. Use the force function,

$$F(t) = 1$$

Note: I suggest to simulate the same problems below but with a load $F(t)$ that takes $t = 1000$ to attain its final value of 1 (final time should also be set to $T_f = 1000$) as an example on how the static solution would look like. Also, you can use $F(t) = \sin(6t)$ and an example of a harmonic load to observe the differences between the two methods. You **do not** need to return results for the two $F(t)$ mentioned here and only $F(t) = 1$ is requested to be analyzed.

Some other parameters that you need for both questions below are:

- Final time $T_f = 10$.
- Time step $\Delta t = 0.01$ (fine time step) and $\Delta t = 0.543$ (coarse time step).
- ICs: Both initial displacement and velocity are zero: $\mathbf{U}(t = 0) = 0, \dot{\mathbf{U}}(t = 0) = 0$.

(a) **Newmark method**: Use the parameters $\alpha = 1/4$ and $\delta = 1/2$. This Corresponds to "Average acceleration", *i.e.,* trapezoidal rule as one of the methods that is a special case of Newmark method.

i. Is the Newmark method with given α, δ conditionally stable or unconditionally stable?

ii. Which mass matrix (consistent or diagonal) will you use with this method and why (answer in one sentence)?

iii. Using equation (259) from the course notes (shown below)

$$\begin{aligned} {}^{t+\Delta t}\dot{\mathbf{U}} &= {}^{t}\dot{\mathbf{U}} + [(1-\delta)\,{}^{t}\ddot{\mathbf{U}} + \delta\,{}^{t+\Delta t}\ddot{\mathbf{U}}]\,\Delta t \\ {}^{t+\Delta t}\mathbf{U} &= {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\,\Delta t + [(\tfrac{1}{2}-\alpha)\,{}^{t}\ddot{\mathbf{U}} + \alpha\,{}^{t+\Delta t}\ddot{\mathbf{U}}]\,\Delta t^2 \end{aligned} \tag{2}$$

and knowing the values of α and δ obtain $\ddot{\mathbf{U}}^{n+1}$ in terms of $\mathbf{U}^{n+1}$ from the second equation, plug it in the first equation to obtain $\dot{\mathbf{U}}^{n+1}$ in terms of $\mathbf{U}^{n+1}$ and finally plug both values in the update equation for t_{n+1},

$$\mathbf{M}\ddot{\mathbf{U}}^{n+1} + \mathbf{C}\dot{\mathbf{U}}^{n+1} + \mathbf{K}\mathbf{U}^{n+1} = \mathbf{R}^{n+1}$$

to obtain,

$$\hat{\mathbf{K}}\mathbf{U}^{n+1} = \hat{\mathbf{R}}, \qquad \text{where} \tag{3a}$$

$$\hat{\mathbf{K}} = a_k\mathbf{K} + a_c\mathbf{C} + a_m\mathbf{M}, \qquad \text{and} \tag{3b}$$

$$\hat{\mathbf{R}} = \mathbf{R}^{n+1} + \mathbf{M}(m_0\mathbf{U}^n + m_1\dot{\mathbf{U}}^n + m_2\ddot{\mathbf{U}}^n) + \mathbf{C}(c_0\mathbf{U}^n + c_1\dot{\mathbf{U}}^n + c_2\ddot{\mathbf{U}}^n) \tag{3c}$$

provide numerical values for $a_k, a_c, a_m, m_0, m_1, m_2, c_0, c_1, c_2$ for the specific values of α, δ given herein. **Do not** directly used the formulas in table 9.4 (pages 302 and 303 of the course notes) to calculate these values. Rather, directly follow from the process described above to compute them.
Note: For this answer use symbolic $\mathbf{M}, \mathbf{K}, \mathbf{C}$. For numerical calculation we use $\mathbf{K}$ and $\mathbf{M}$ from (1) (with lumped or consistent mass matrix used based on your answer from question 2 and damping matrix $\mathbf{C} = \mathbf{0}$).

(b) **Central difference method**: Use the scheme §4.3.1 and equations (244) and (255) to solve the 2 dof problem with provided system matrices, time step, and final time.

i. Is this scheme explicit or implicit?

ii. If the central difference method is explicit name an implicit LMS method and if implicit name an explicit LMS method for structural dynamics.

iii. Is it conditionally stable or unconditionally stable?

iv. What mass matrix should be used with central difference method (consistent or diagonal)? Provide at least **two reasons** for your answer, including comments on the efficiency of the method **and** which one is more appropriate with this method in terms of handling **period elongation error (*i.e.*, frequency error)**.

v. If the method is conditionally stable what will be the absolute maximum time step we can take with this method if consistent or lumped mass matrices are used. Use frequencies from global system to answer this question (that is questions 1(b)i and 1(b)iii above). Label these two time steps as Δt_{c_M} and Δt_{d_M}.
Note: Note that the maximum time step of central difference method is not simply $1/\omega_m$ with ω_m being a representative maximum frequency. It may be a factor of it (obviously if it is conditionally stable). You need to refer to the course notes to find the factor in front of $1/\omega_m$ (if any).

vi. Based on the answer to the previous question, which mass matrix will require a more stringent time step (if any). That is which one of Δt_{c_M} and Δt_{d_M} is smaller?

vii. Now given that we often do not find the actual frequencies of a structure, use the maximum element frequencies (from item 1(b)ii and 1(b)iv) to compute maximum time steps $\Delta t^e_{c_M}$ and $\Delta t^e_{d_M}$ in which instead of the correct MDOF system maximum frequency we use element level maximum frequencies (from item 1(b)ii and 1(b)iv). Is this approach conservative (That is $\Delta t^e_{c_M} < \Delta t_{c_M}$ and $\Delta t^e_{d_M} < \Delta t_{d_M}$)? What are the values of $\Delta t^e_{c_M}$ and $\Delta t^2_{d_M}$?

(c) Numerically solve both **Newmark method** (with given parameters α, δ) and **central difference** method for the given $\mathbf{K}, \mathbf{M}$ and $\mathbf{R}$ (1), Δt, T_f and $F(t)$, and ICs. For each method choose its appropriate mass matrix (consistent or lumped) from (1). Deliverables are,

i. Plots of the free end point displacement ($U_2(t)$) (y axis) versus time (x axis) for **both Newmark and central difference methods** for $t = 0$ to T_f and **both time steps** Δt provided.

ii. In terms of the results you obtain with the two values of Δt explain the type of solutions you observe? What type of problems we can encounter by using large time steps with conditionally stable and unconditionally stable methods?

iii. Source code(s) (Matlab, Mathematica, C++, *etc.*) used for your computations.

3. ($(5(a) + 90(b, c) + 5(d) =$) **100 Points**) **Runge-Kutta method**: Solve the SDOF initial value problem,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) = yt^3 - 1.5y \tag{4}$$

for $t = 0$ to $T_f = 2$ and IC $y(0) = 1$. Display all your results on the same plot.

- Analytically.
- Forward Euler method with $\Delta t = 0.5$ and $\Delta t = 0.125$.
- Midpoint method with $\Delta t = 0.5$.
- Fourth-order explicit RK method (RK4) with $\Delta t = 0.5$.

Deliverables are,

(a) Derivation of the analytical solution.

(b) A plot showing all the solutions mentioned above.

(c) Source code(s) used for your computations.

(d) Between Euler method with $\Delta t = 0.125$ and RK4 method with $\Delta t = 0.5$ both have the same "stop points" while advancing the solution. With the former they are time steps and in the latter one in every four is a time step and the other three are stages. Which one is more accurate? Explain why?

Hint: It will be much easier to computer code the equation (262) (shown below) for general a, b, c, s:

$$y_{n+1} = y_h + \Delta t \sum_{i=1}^{s} b_i k_i \qquad \text{where} \tag{5a}$$

$$k_i = f(t_n + \Delta t c_i, y_n + \Delta t \sum_{j=1}^{i-1} a_{ij} k_j), \qquad 1 \leq i \leq s \tag{5b}$$

All needed to be done is provide the values for vectors b, c and matrix a in the computer code and switch based on the value of s (and potentially options within a given s such as Heun and Midpoint methods for $s = 2$) in your code. All methods mentioned (forward Euler, Midpoint, and RK4) are RK methods with different s, a, b, c. You can very easily code a general purpose RK method that has s, Δt, T_f and the function f (or an option number for the coded functions f) as input arguments to your program.