1. $((10(a)+50(b)=)$ **60 Points**) **Simple von Neumann analysis**: Stability analysis of forward-time central-space (FTCS) method.

   (a) Using the FD equation (27c)

   $$\frac{v_m^{n+1}-v_m^n}{k}+a\frac{v_{m+1}^n-v_{m-1}^n}{2h}=0$$

   for the advection equation,

   $$u_{,t}+au_{,x}=0 \tag{1}$$

   show that the update equation is,

   $$v_m^{n+1}=v_m^n+\frac{\bar{k}}{2}v_{m-1}^n-\frac{\bar{k}}{2}v_{m+1}^n \quad \text{for} \tag{2a}$$

   $$\bar{k}=\frac{ka}{h} \qquad \text{Normalized time step for advection equation} \tag{2b}$$

   (b) Using von-Neumann analysis and by using (448),

   $$v_{m+a}^{n+b}=e^{\imath a\theta}g^b v_m^n \qquad \text{Temporally one-step FD stencils} \tag{3}$$

   for this one step method show that the amplification factor $g>1$ for all $\bar{k}$ so the method is unconditionally unstable.

2. $((10(a)+10(b)+10(c)+20(d)=)$ **50 Points**) **PDE mode transition and design of a FD stencil**: Consider the following PDE from (4),

   $$\tau u_{,tt}+u_{,t}-Du_{,xx}=0 \qquad \text{1D relaxed diffusion equation} \tag{4}$$

   where $\tau\geq 0, D>0$. This PDE is hyperbolic for $\tau>0$ and if $\tau=0$ the equation is a simple diffusion equation whose response is characterized by damping and diffusion solution u. On the other hand if we do not have $u_{,t}$ term the PDE is $\tau u_{,tt}-Du_{,xx}=0$ which is a wave equation which splits and propagates IC on value $(u_0(x))$ to the left and right with speeds $\pm a$, $a=\sqrt{\frac{D}{\tau}}$ (*cf.* D'Alembert solution (591)). That is the response of these systems are vastly different. If appropriate explicit FD schemes are used their corresponding maximum time steps are proportional to D/h^2 and c/h respectively.

   Limiting cases for the PDE (4): Using dimensional analysis and discussion in §7.3 describe that for length scale $\tilde{L}$,

   $$\tilde{L}=\sqrt{\tau D} \tag{5}$$

   we have the following two limiting cases for equation (4)

   $$\tilde{L}_0\ll\tilde{L} \qquad u_{,tt}-a^2u_{,xx}=0 \qquad k_{\max}\propto\frac{h}{a} \qquad \text{Undamped hyperbolic limit} \tag{6a}$$

   $$\tilde{L}_0\gg\tilde{L} \qquad u_{,t}-Du_{,xx}=0 \qquad k_{\max}\propto\frac{h^2}{D} \qquad \text{Diffusion (parabolic) limit(??)} \tag{6b}$$

   where

   $$a=\sqrt{\frac{D}{\tau}} \qquad \text{is the wave speed} \tag{7}$$

   , $\tilde{L}_0$ is a length scale of interest, *e.g.*, observation length scale (*i.e.*, element length h or length scale relevant to a particular problem considered) and $k_{\max}$ is the maximum time step of an explicit method.

   Note: For the dimensional analysis provide response for the following three items:

   (a) The length scale implied by the PDE (4) is $\tilde{L}=\sqrt{\tau D}$.

   (b) The time step limit for (6a) (if the scheme is conditionally stable with a maximum time step $k_{\max}$) is proportional to $\frac{h}{a}$.

   (c) The time step limit for $k_{\max}$ in (**??**) (again if a maximum time step $k_{\max}$ exists) is proportional to $\frac{h^2}{D}$.

   In the dimensional analysis you need to use the scales of parameters involved. For example, $[\tau]=T, [a]=L/T$ where [.] is the physical dimension of a quantity and L, T are length and time respectively.

   Fourth item (d): For the explanation of why for small length (and time) scales the undamped hyperbolic limit is apporached and why for large ones the diffusion limit is approached you can refer to §7.3 and equations (570b), (571), and (572) (particularly (572b)). Be **very brief (less than 4-5 sentences) in your explanation**.

3. $(((25+25+3\times 10)(a)+(25+25+4\times 10)(b)=)$ **170 Points) FD formulation for a problem of the form** (4): We want to formulate an appropriate explicitly FD formulation for (4). The schemes considered are both consistent so the proof of stability will be sufficient for establishing their convergence. The stability analysis also provides the maximum time step $k_{\max}$ which is not clear what it would be for a problem of the type (4).

   The stability analysis for both cases involves von Neumann analysis which plugs (445),

$$v_m^n = e^{\imath m\theta}\hat{v}^n \tag{8}$$

   in the FD stencil. For both methods considered we obtain an equation for amplification factor g ($\hat{v}^{n+1} = g\hat{v}^n$) in the form,

$$g^2 - 2A_1 g + A_2 = 0 \tag{9}$$

   where coefficients A_1 and A_2 will be obtained based on the von Neumman analysis (*i.e.,* insertion of (8) in the FD stencil).

   If the coefficients A_1 and A_2 are real (which will be the case for the examples in this HW), the condition $g \le 1$ is equivalent to,

$$-1 \le A_2 \le 1, \qquad -\frac{A_2+1}{2} \le A_1 \le \frac{A_2+1}{2} \tag{10}$$

   Side Note (FYI): Equation (10) is basically the same as (362) but allowing the case $|A_1| = A_2 = 1$ given that we allow repeated root of $g_1 = g_2 = \pm 1$ since for a hyperbolic equation (4) growth in the form $\hat{v}^{n+1} = (\pm 1)t\hat{v}^n$ is allowed; *cf.* §6.4.2 (480) and (487) for further discussion. Finally, since in all the problems considered in this assignment g does not explicitly depend on k the simpler stability condition $|g| \le 1$ is used rather than (432) $|g| < 1 + Kk$ for one-step methods and its generalization for the two step methods herein.

   Below we consider two different stencils for the solution of (4). By your analysis you will observe that one is more appropriate for the solution of this PDE.

   (a) **CCC scheme**: We use central difference for $u_{,tt}$, $u_{,t}$, and $u_{,xx}$ in (4) to obtain,

$$\tau\frac{v_m^{n+1} - 2v_m^n + v_m^{n-1}}{k^2} + \frac{v_m^{n+1} - v_m^{n-1}}{2k} - D\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2} = 0 \tag{11}$$

   Demonstrate the following:

   i. By von Neumann analysis (*i.e.,* plugging $v_m^n = e^{\imath m\theta}\hat{v}^n$ from (8)) demonstrate

$$g^2 - 2A_1 g + A_2 = 0 \qquad \text{where} \qquad \begin{cases} A_1 = \frac{2\bar{\tau} - 4\bar{\bar{k}}\sin^2\left(\frac{\theta}{2}\right)}{2\bar{\tau}+1} \\ A_2 = \frac{2\bar{\tau}-1}{2\bar{\tau}+1} \end{cases} \tag{12a}$$

$$\bar{\bar{k}} = \frac{kD}{h^2} \qquad \text{normalized time step for parabolic PDE } u_{,t} - Du_{,xx} = 0 \tag{12b}$$

$$\bar{\tau} = \frac{\tau}{k} \qquad \text{normalized } \tau \text{ by } k \tag{12c}$$

   ii. Using (12a) and the condition (10) show that the stable time step for the scheme (11) is,

$$k \le k_{\max}, \quad \text{for} \quad k_{\max} = \frac{h}{a}, \quad \text{where from (7)} \quad a = \sqrt{\frac{D}{\tau}}, \quad \text{that is} \tag{13a}$$

$$\bar{k} \le 1, \quad \text{where} \tag{13b}$$

$$\bar{k} = \frac{ka}{h}, \qquad \text{normalized time step for the wave equation } u_{,tt} - a^2 u_{,xx} = 0 \tag{13c}$$

   Note: Clearly, if you cannot derive (12a) you can still use it and (10) to demonstrate (17).

   iii. Compare the stability condition (17) with that of the central time central space stencil for the undamped hyperbolic equation $u_{,tt} - a^2 u_{,xx} = 0$ ($a = \sqrt{D/\tau}$) which is discussed in §6.4.2.

   iv. Based on your answer from previous item, does the term $u_{,t}$ in (4) (discretized by central time term $\frac{v_m^{n+1} - v_m^{n-1}}{2k}$ in (11)) affect the stability of $\tau u_{,tt} + u_{,t} - Du_{,xx} = 0$ in the FD scheme (11) ? Later, you will comment whether this behavior is favorable or not.

   v. Now, let us focus on another feature of (11) when $\tau = 0$, that is when we solve the diffusion equation $u_{,t} - Du_{,xx} = 0$. Your analysis from (12) and (17) still holds for this case, with the difference that $g_1 = g_2 = \pm 1$ is not acceptable anymore given that for this temporally first order PDE (similar to any other temporally first order PDE) FD growth in the form $\hat{v}^{n+1} = (\pm 1)t\hat{v}^n$ is not permitted; *cf.* the analysis of leapfrog method in §6.4.1.

In any case, apart from this minor point (that $g_1 = g_2 = \pm 1$ is not acceptable for $\tau = 0$) show that (11) for $\tau = 0$ is **unconditionally unstable**. That is, the discretization

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} - D\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2} = 0 \tag{14}$$

for $u_{,t} - Du_{,xx} = 0$ is unconditionally unstable.
Hint: As mentioned above, the analysis above ((12) and (17)) for general τ would carry for $\tau = 0$. So, you do not need to do the von Neumann analysis for (15) from the beginning.

(b) **CFC scheme**: We use central difference for $u_{,tt}$ and $u_{,xx}$ and forward difference for $u_{,t}$ in (4) to obtain,

$$\tau\frac{v_m^{n+1} - 2v_m^n + v_m^{n-1}}{k^2} + \frac{v_m^{n+1} - v_m^n}{k} - D\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2} = 0 \tag{15}$$

Demonstrate the following:

i. By von Neumann analysis (*i.e.*, plugging $v_m^n = e^{\imath m\theta}\hat{v}^n$ from (8)) demonstrate

$$g^2 - 2A_1 g + A_2 = 0 \qquad \text{where} \qquad \begin{cases} A_1 = \frac{\bar{\tau} + \frac{1}{2} - 2\bar{\bar{k}}\sin^2\left(\frac{\theta}{2}\right)}{\bar{\tau}+1} \\ A_2 = \frac{\bar{\tau}}{\bar{\tau}+1} \end{cases} \tag{16}$$

where as in (12) $\bar{\bar{k}} = \frac{kD}{h^2}$ and $\bar{\tau} = \frac{\tau}{k}$.

ii. Using (16) and the condition (10) show that the stable time step for the scheme (15) is,

$$k \le k_{\max}, \quad \text{for} \quad k_{\max} = \frac{h^2}{2D}\left(\frac{1 + \sqrt{1 + \left(4\frac{\tilde{L}}{h}\right)^2}}{2}\right) \tag{17a}$$

recall $\tilde{L} = \sqrt{\tau D}$ from (5).
Hint: The application of (10) on (16) yields $\bar{\bar{k}} \le \bar{\tau} + \frac{1}{2}$ whose solution results in (17). Also again you can directly proceed from (16) if you fail to obtain the values A_1, A_2 from (15).

iii. **Very small grid size limit**: Consider the limiting case $\frac{h}{\tilde{L}} \ll 1$, *i.e.*, a very small grid size relative to the length scale $\tilde{L}$ for the PDE (4). What time step we obtain from (17) and does it match what is expected from (6a) for the undamped hyperbolic $\tau u_{,tt} - Du_{,xx} = 0$ limit of (4) ($\tau u_{,tt} + u_{,t} - Du_{,xx} = 0$) for $h(= \tilde{L}_0) \ll \tilde{L}$.

iv. **Very large grid size limit**: Consider the limiting case $\frac{h}{\tilde{L}} \gg 1$, *i.e.*, a very large grid size relative to the length scale $\tilde{L}$ for the PDE (4). What time step we obtain from (17) and does it match what is expected from (**??**) for the diffusion equation $u_{,t} - Du_{,xx} = 0$ limit of (4) ($\tau u_{,tt} + u_{,t} - Du_{,xx} = 0$) for $h(= \tilde{L}_0) \gg \tilde{L}$.

v. In light of the previous two answers, compare the efficiency of CCC scheme (11) and CFC scheme (15) for the solution of (4) ($\tau u_{,tt} + u_{,t} - Du_{,xx} = 0$) in the limit $h \gg \tilde{L}$. Which scheme gives smaller time step which happens to also be consistent with the physics of the problem from (**??**)?

vi. Based on all previous questions which scheme (CCC or CFC) would you use for the solution of (4)?

Side Note (FYI): The solution of problems that have different limiting PDEs as the relevant length/time scales vary (*i.e.*, grid space and time steps) is a very active research topic. For example the relaxed advection-diffusion-reaction problem $\tau u_{,tt} + u_{,t} + vu_{,x} - Du_{,xx} = -ru$ (v is advection speed and r the reaction rate) shows several PDE mode transitions. The design of numerical methods that can consistently solve all limiting cases in a unified manner is a challenging task. For more information refer to the discussion on "asymptotic preserving" schemes in [Jin, 2010] (shared with you in dropbox folder Dynamics of continua/Books_courseNotes/StiffSystems/Relaxation).

4. ((25(*a*) + 25(*b*) + 40(*c*) + 15(*d*) + 15(*e*) + 20(*f*, **extra credit**) =) **120 + 20 e.c. Points**) **An unconditionally stable explicit method (with conditional consistency)!:** FD schemes that can be explicitly solved and have no time step constraint are not encountered except in special cases in 1D where the update of an implicit method can be done explicitly; *cf.* BTBS scheme applied to 1D advection equation $u_{,t} + au_{,x} = 0$ in §2.1.9 as one example. Genuinely explicit methods often have a time step limit imposed by stability constraint. Below, we show an explicit method that does not have any stability limits but its conditional consistency instead limits its time step.

Consider the diffusion equation,

$$u_{,t} - Du_{,xx} = 0 \tag{18}$$

solved with Dufort-Frankel FD method,

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} - D\frac{v_{m+1}^n - (v_m^{n-1} + v_m^{n+1}) + v_{m-1}^n}{h^2} = 0 \tag{19}$$

which is basically similar to (15) but $-2v_m^n$ in the stencil for $u_{,xx}$ being replaced by $(v_m^{n-1} + v_m^{n+1})$.

(a) **von Neuman stability analysis**: By the insertion of (8) in the FD stencil (19) show that the equation for g is,

$$g^2 - 2A_1 g + A_2 = 0 \qquad \text{where} \qquad \begin{cases} A_1 = \frac{2\bar{\bar{k}}\cos\theta}{2\bar{\bar{k}}+1} \\ A_2 = \frac{2\bar{\bar{k}}-1}{2\bar{\bar{k}}+1} \end{cases} \tag{20}$$

(b) By applying (10) to (20) (the equation (20) cannot have repeated roots $g_1 = g_2 = \pm 1$ so we do not need to worry about such repeated roots for the temporally first order ODE (18)) demonstrate that the explicit method of Dufort-Frankel is **unconditionally stable!**

(c) **Conditional consistency**: By plugging Taylor series expansion of terms in (19) for an infinitely smooth function ϕ, *e.g.*, $\phi_m^{n+1} = \phi_m^n + k\dot{\phi} + \sum_{i=2}^{\infty} \frac{k^i}{i!}\frac{\partial^i \phi}{\partial t^i}$, $\phi_{m+1}^n = \phi_m^n + h\phi_{,x} + \sum_{i=2}^{\infty} \frac{h^i}{i!}\frac{\partial^i \phi}{\partial x^i}$, form numerical PDE operator $P_{h,k}\phi$. Then separate the exact PDE operator $P\phi = u_{,t} - Du_{,xx}$ and show,

$$P_{h,k}\phi - P\phi = \frac{1}{6}k^2\frac{\partial^3\phi}{\partial t^3} - h^2\frac{D}{12}\frac{\partial^4\phi}{\partial x^4} + \frac{k^2}{h^2}D\ddot{\phi} + \text{H.O.T.} \tag{21}$$

(d) From (21) show that (19) is conditionally consistent. That is for this scheme $P_{h,k}\phi - P\phi \to 0$ when $h, k \to 0$ only **when k tends to zero faster than h.**

(e) Inconsistency of a FD scheme basically means that we are solving another PDE. From (21) it is evident that if $D\frac{k^2}{h^2}$ is bounded and not tending to zero; *e.g.*, $k \propto h$ as for an explicit scheme for a hyperbolic PDE, we basically solve $D\frac{k^2}{h^2}u_{,tt} + u_{,t} - Du_{,xx}$. This can be demonstrated in another way too; we rewrite (19) as,

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} - D\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2} + \text{ other terms} = 0 \tag{22}$$

Show that the "other terms" actually correspond to $D\frac{k^2}{h^2}u_{,tt}$. Clearly the first two terms are the FD stencils for $u_{,t}$ and $Du_{,xx}$.

Side Note (FYI): In fact, many FD schemes basically add other differential terms to the PDE that for a consistent scheme they must tend to zero as $h, k \to 0$. For example, in the discussion of FV solution for elastodynamic problem in §2.2.6.2 we discussed that the use of Riemann fluxes effectively adds a numerical diffusion coefficient (116) $D_h = \frac{hc}{2}$ whose value (and its contribution to the PDE) vanish as grid size $h \to 0$. For this problem, the relaxation term $D\frac{k^2}{h^2}u_{,tt}$ only vanishes if the scheme is consistent; *i.e.*, in this case k tending to zero faster than h.

(f) **Relation to CCC scheme above (extra credit, do not need to return)**: As discussed in previous question Dufort-Frankel is basically solving $D\frac{k^2}{h^2}u_{,tt} + u_{,t} - Du_{,xx} = 0$. A closer examination of what you obtain from (22) (*i.e.*, when other terms are determined) shows that this equation is identical to CCC scheme (11) for $\tau = \frac{Dk^2}{h^2}$. Show that the time step constraint (17) is basically trivially satisfied.

Side Note (FYI): This is another confirmation that Dufort-Frankel method is unconditionally stable, but obviously when $\tau = \frac{Dk^2}{h^2}$ does not tend to zero when $h, k \to 0$ we have a stable method that converges to the solution to a relaxed diffusion equation rather than the underlying diffusion equation (18) ($u_{,t} - Du_{,xx} = 0$). Sometimes detection of inconsistencies of numerical methods can be difficult since the solution does not blow up as with strongly unstable method and we may get reasonably well-behaved solutions but with the solutions corresponding to another PDE!

Side Note (FYI): As another interesting feature of Dufort-Frankel we observe all needed for consistency is that k tend to zero faster than h. That is, $k/h \to 0$ as $h, k \to 0$. Clearly, typical time step of explicit methods for diffusion equation (18) ($u_{,t} - Du_{,xx} = 0$) requires $k < h^2/D$. So, a scaling of the form $k \propto h^2/D$ makes Dufort-Frankel scheme consistent.

What can be achieved beyond this is that any time step scaling with $k/h \to 0$ works for Dufort-Frankel scheme. For example, we can have $k \propto h^{1.5}$ or even $k \propto h^{1.001}$ and still have a consistent scheme with much less stringent time step than a scaling of the form $k \propto h^2/D$ which is typical for explicit solvers of diffusion equation. Clearly, the slower the relaxation time $\tau = \frac{Dk^2}{h^2}$ tends to zero the slower the convergence of FD solution will be to the exact solution but again the interesting feature is that such scalings would result in a convergent scheme while typcially for explicit schemes applied to diffusion equation $k \propto h^2$ is required for convergence.