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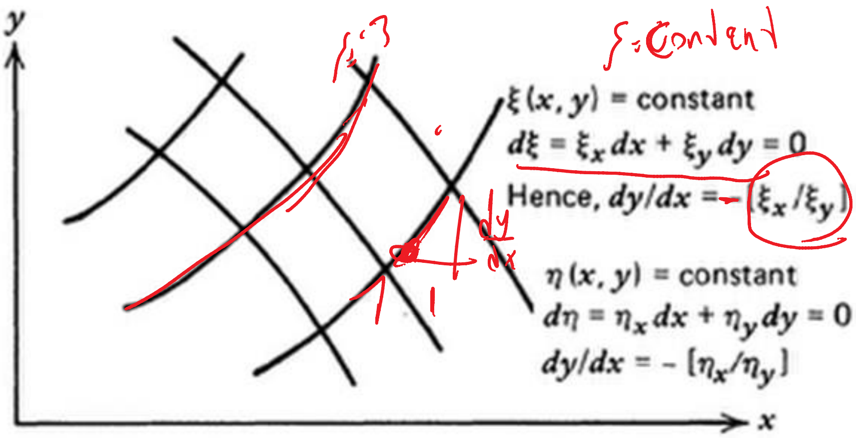
2nd order PDE in terms of 2 independent parameters



$$[\xi_x/\xi_y] = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

Example:

- A constant coefficient hyperbolic example:

$$u_{xx} - 4u_{yy} + u_x = 0$$

$$\frac{dy}{dx} = -[\xi_x/\xi_y] = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -2$$

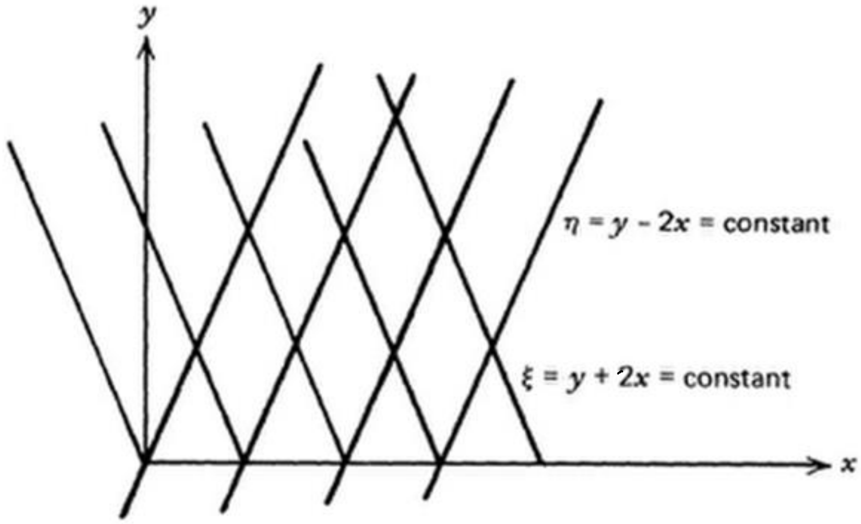
$$\frac{dy}{dx} = -[\eta_x/\eta_y] = \frac{B + \sqrt{B^2 - 4AC}}{2A} = 2$$

$\boxed{y = -2x + C} \Rightarrow y + 2x = C = \xi$

$\frac{dy}{dx} = -2$ on ξ is constant

$\frac{dy}{dx} = 2$ on η is constant

$y = 2x + C' \Rightarrow y - 2x = C' = \eta$



$u_{\xi\eta}$ + LOTS = 0

lower order terms = 0

$\int u_{\xi\eta}\, d\xi + \int \text{LOTS}\, d\xi = 0$

if

we can do the integrals

u_η + " = 0

$\int u_\eta\, d\eta + \int$ " $= 0$

$B u_{xy}$ $B = 0$

- Consider the PDE

$$A\; y^2 u_{xx} - x^2 u_{yy} = 0 \quad x > 0 \quad y > 0$$

type of PDE $B^2 - 4AC = 4x^2y^2 > 0$

hyperbolic for all $|x|>0$ & $|y|>0$

Equations for characteristics:

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = -\frac{x}{y}$$

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{x}{y}$$

$\xi = y^2 - x^2$

$\frac{dy}{dx} = -\frac{x}{y}$

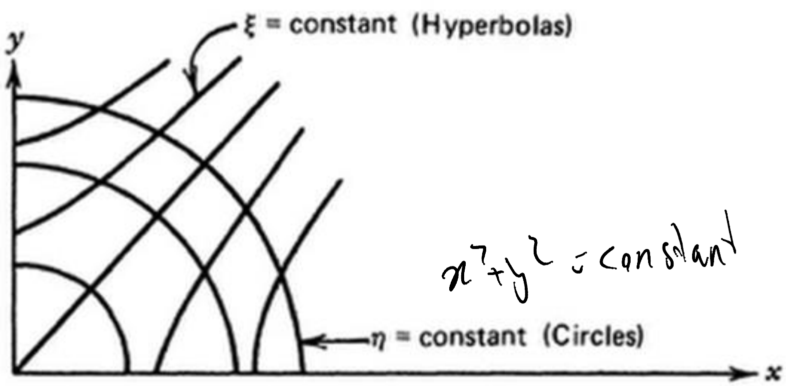
$y\,dy = -x\,dx$

$y^2 = -x^2 + c$

$\Rightarrow c = y^2 + x^2$

η

$x^2 - y^2 =$ constant hyperbolas



$x^2 + y^2 =$ constant

2nd order PDEs with more than 2 independent variables:

$$1\, u_{,tt} - \overset{-c^2}{c^2}\, u_{,xx} - \overset{-c^2}{c^2}\, u_{,yy} = 0$$

wave eqn in 2D

$$\begin{bmatrix} \partial/\partial t & \partial/\partial x & \partial/\partial y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -c^2 & 0 \\ 0 & 0 & -c^2 \end{bmatrix} \begin{bmatrix} \partial/\partial t \\ \partial/\partial x \\ \partial/\partial y \end{bmatrix} u = 0.$$

$$\left[\frac{\partial}{\partial t} \ \frac{\partial}{\partial x} \ \frac{\partial}{\partial y}\right] \begin{bmatrix} \cdots & & \\ 0 & -c & \\ 0 & 0 & -c^2 \end{bmatrix} \begin{bmatrix} \cdots \\ \partial/\partial x \\ \partial/\partial y \end{bmatrix} u = 0.$$

eqn values all > 0 except one < 0

OR " < 0 except " > 0

$$1\, u_{,xx} + 1\, u_{,yy} + 1\, u_{,zz} = 0$$

Elliptic

3D Poisson eqn

$$[\nabla]^T \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} [\nabla] u = 0$$

eigenvalues all > 0

OR " < 0

$$0\, u_{,tt} \overset{-k}{-k u_{,xx}} \overset{-k}{-k u_{,yy}} + c u_{,t} = 0$$

Parabolic

heat eqn in 2D

$$[\nabla]^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & -k & 0 \\ 0 & 0 & -k \end{bmatrix} [\nabla] u + c u_{,t} = 0$$

0 eigenvalue

- (H) for ($Z = 0$ and $P = 1$) or ($Z = 0$ and $P = n - 1$)
- (P) for $Z > 0$ ($\Leftrightarrow \det \mathbf{a} = 0$)
- (E) for ($Z = 0$ and $P = n$) or ($Z = 0$ and $P = 0$)
- (ultraH) for ($Z = 0$ and $1 < P < n - 1$)

where

- Z: nb. of zero eigenvalues of $\mathbf{a}$
- P: nb. of strictly positive eigenvalues of $\mathbf{a}$

$$A u_{,tt} + B u_{,xt} + C u_{,xx} + u_x = 0$$

$$\begin{bmatrix} \partial/\partial t & \partial/\partial x \end{bmatrix} \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} \partial/\partial x & \partial/\partial x \end{bmatrix} + u_x = 0$$

make it symmetric
for off-diagonal values $\Big\} \Rightarrow$ it has n Real eigenvalues

$$\det \begin{bmatrix} A - \lambda & B/2 \\ B/2 & C - \lambda \end{bmatrix} = 0$$

$$\lambda^2 - (A + C)\lambda + AC - \frac{B^2}{4} = 0$$

$$\lambda^2 - (A+C)\lambda - \frac{1}{4}(B^2 - 4AC) = 0$$

		$B^2 - 4AC$	
– hypolic if	$\lambda_1 \lambda_2 < 0$	> 0	matches our previous classification
– elliptic "	$\lambda_1 \lambda_2 > 0$	< 0	
– parabolic	$\lambda_1 \lambda_2 = 0$	$= 0$	

$$\lambda_1 \lambda_2 = \underbrace{-\frac{1}{4}}(B^2 - 4AC)$$

$$ax^2 + bx + c = 0 \qquad x_1 x_2 = \frac{c}{a} \qquad x_1 + x_2 = \frac{-b}{a}$$

D'Alembert solution of the wave equation

PDE $\quad u_{tt} = c^2 u_{xx} \quad -\infty < x < \infty \quad 0 < t < \infty$

ICs $\quad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \quad -\infty < x < \infty$

The solution is,

$$u(x,t) = \frac{1}{2}\left[f(x - ct) + f(x + ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

- The characteristic parameters $\xi = x + ct$, $\eta = x - ct$

  cast the PDE into its canonical form

$$u_{tt} - c^2 u_{xx} = 0 \Rightarrow \boxed{u_{\xi\eta} = 0}$$

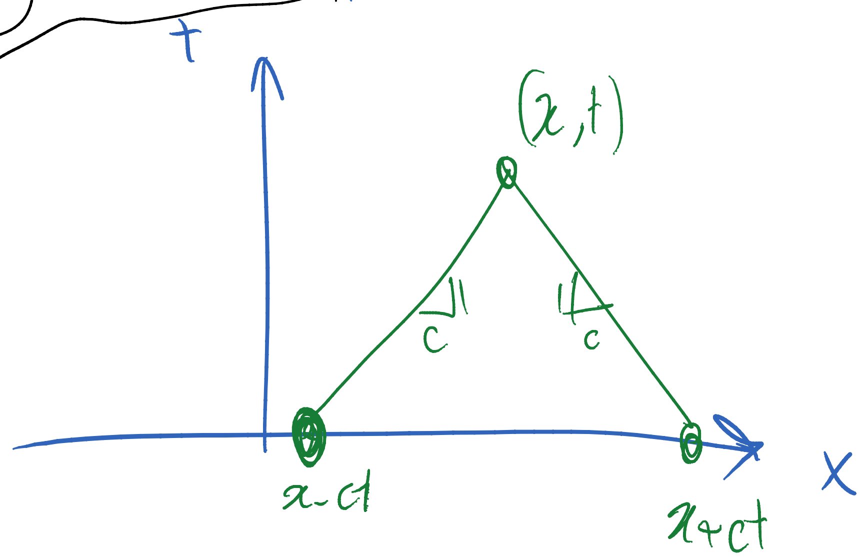
$$u(\xi,\eta) = \Phi(\eta) + \psi(\xi)$$

$$u(\xi,\eta) = \phi(x-ct) + \psi(x+ct)$$

$$u(x,t) = \phi(x-ct) + \psi(x+ct)$$

still unknown

we find ϕ, ψ by ICs



ICs $\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases} \quad -\infty < x < \infty$ $\Rightarrow$

$$u(\xi,\eta) = \Phi(\eta) + \psi(\xi)$$

$$u(\xi,\eta) = \Phi(\eta) + \psi(\xi)$$

$$\phi(x) + \psi(x) = f(x)$$
$$-c\phi'(x) + c\psi'(x) = g(x)$$

- By integrating the second equation we get

$$\phi(x) + \psi(x) = f(x)$$
$$-c\phi(x) + c\psi(x) = \int_{x_0}^{x} g(\xi)\, d\xi + K$$

$$\phi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\xi)\, d\xi$$
$$\psi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\xi)\, d\xi$$

$$u(\xi,\eta) = \Phi(\eta) + \psi(\xi) = \varphi(x - ct) + \psi(x + ct)$$

Right-going wave (speed c) Left-going wave (speed c)

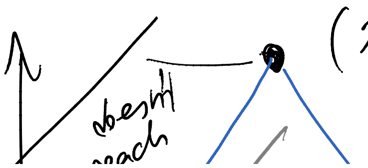
$$\boxed{u(x,t) = \frac{1}{2}[f(x - ct) + f(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi}$$

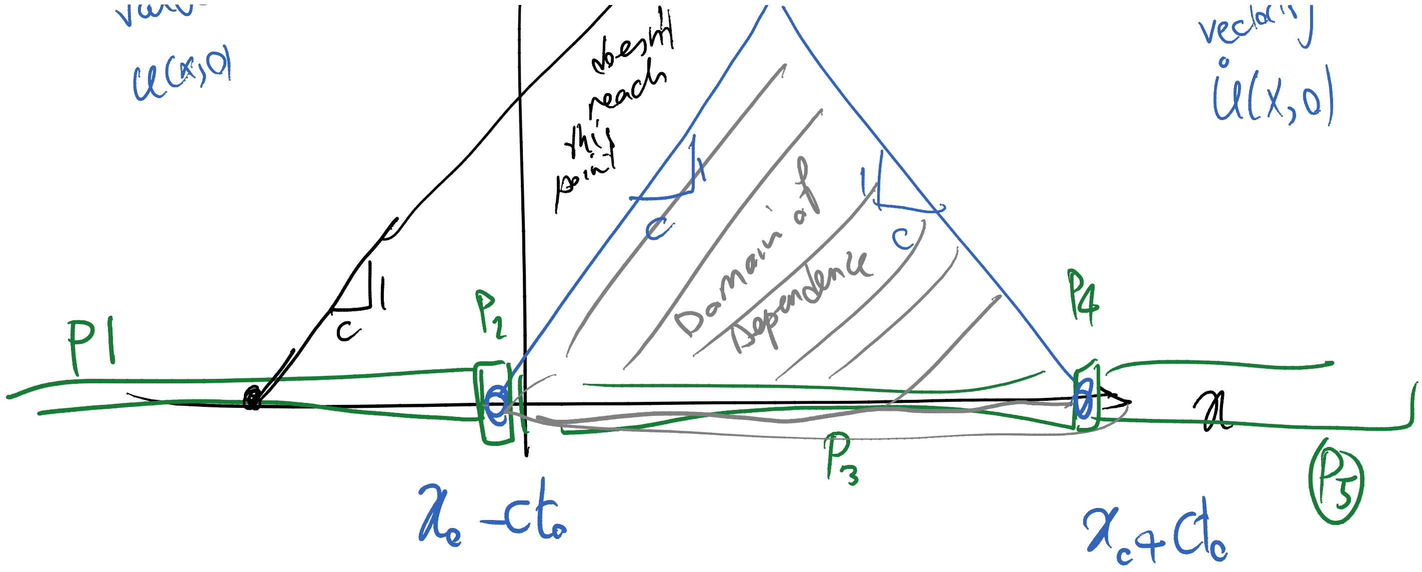
$$\boxed{u(x,t) = \frac{1}{2}[f(x_0 - ct_0) + f(x_0 + ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(\xi)\, d\xi}$$

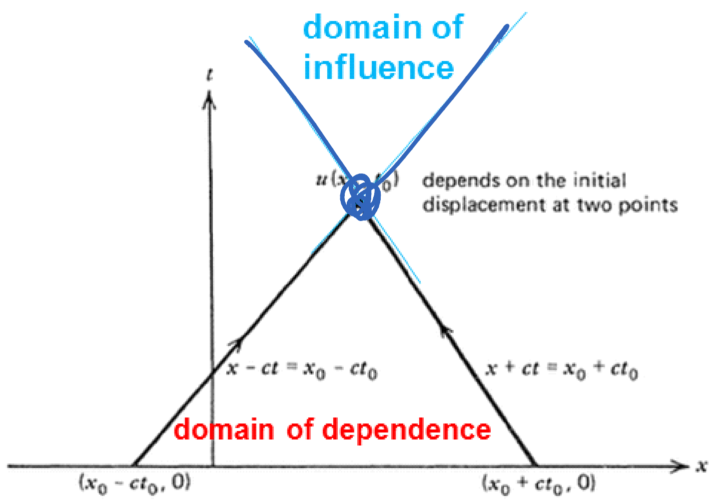
initial value $u(x,0)$

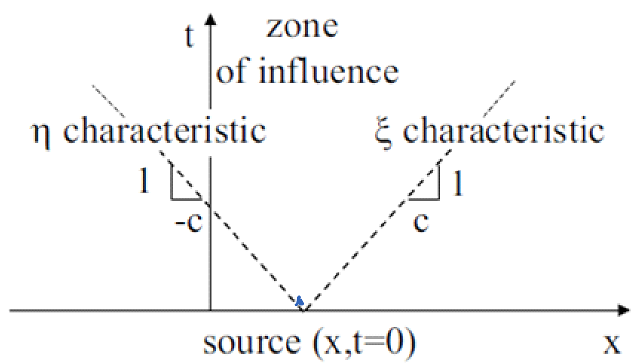
initial velocity

t (x_0, t_0)









$$u_{,tt} - c^2 u_{,xx} = 0$$

$$V = u_{,t} \qquad \varepsilon = u_{,x}$$

$$\dot{V} - c^2 \varepsilon_{,x} = 0$$

$$\dot{\varepsilon} - V_{,x} = 0 \qquad \dot{\varepsilon} = \dot{(u_{,x})} = (\dot{u})_{,x} = V_{,x}$$

$$\mathring{\varepsilon} - V_{,x} = 0 \qquad \dot{\varepsilon} = (\dot{u_{,x}}) = (\dot{u})_{,x} = V_{,x}$$

$$\begin{bmatrix} V \\ \varepsilon \end{bmatrix}_{,t} + \begin{bmatrix} 0 & -c^2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} V \\ \varepsilon \end{bmatrix}_{,x} = 0$$

- Assume we want to solve the system of semi-linear first order PDEs,

$$\text{PDE :} \qquad \mathbf{q}_{,t} + \mathbf{A}\mathbf{q}_{,x} = \mathbf{s}(\mathbf{q}, x, t)$$

$$\text{IC :} \qquad \mathbf{q}(x, 0) = \mathbf{q}_0(x)$$

where

$$\mathbf{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \qquad \text{vector of unknown fields}$$

$$\mathbf{A}, \qquad n \times n \text{ spatial flux matrix}$$

$$\mathbf{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix} \qquad \text{source term (can be nonlinear in } \mathbf{q}$$

$$n \qquad \text{number of fields}$$

$$\dot{q} + A q_{,x} = 0$$

$$\boxed{\dot{q}_1 + A_{11} q_{1,x}} + A_{12} q_{2,x} + \dots + A_{1n} q_{n,x} = 0$$

$$\dot{q}_2 + A_{21} q_{1,x} + A_{22} q_{2,x} + A_{2n} q_{n,x} = 0$$

$$\boxed{\dot{q}_1 + A_{11} q_{1,x}} + A_{12} q_{2,x} + \dots + A_{1n} q_{n,x} = 0$$

$$\dot{q}_2 + A_{21} q_{1,x} + A_{22} q_{2,x} + A_{2n} q_{n,x} = 0$$

$$\vdots$$

invertable matrix

$$L A = \Lambda L$$

diagonal

$$Lx \quad q_{,t} + A q_{,x} = S$$

$$(Lq)_{,t} + \underbrace{(LA)\, q}_{,x} = LS$$

$$(Lq)_{,t} + \Lambda (Lq)_{,x} = LS$$

Diagonal

$$\left\{\begin{array}{l} w_{,t} + \Lambda w_{,x} = S^w \qquad S^w = LS \\ w_{i,t} + \Lambda_{ii} w_{i,x} = S^w_i \\ w(x,0) = Lq(x,0) = Lq_0(x) \end{array}\right.$$

$$w = Lq$$

characteristic variables

$\Downarrow$ characteristic variables

$$\omega(x,t)$$
$$\Downarrow$$
$$q(x,t) = L^{-1}\,\omega(x,t)$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix} = \begin{bmatrix} L_{11} & & L_{1n} \\ & & \\ L_{n1} & & L_{nn} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$L A = \Lambda L$$

$$\begin{bmatrix} l_1 \\ l_2 \\ l_i \\ \vdots \\ l_n \end{bmatrix} A = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_i & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} l_1 \\ \vdots \\ l_i \\ \vdots \\ l_n \end{bmatrix}$$

row vector $\vec{l_i}\, A = \lambda_i \vec{l_i}$ no summation on i

Left eigenvector of A

transpos

$$(A^T)\, l_i^T = \lambda_i\, \vec{l}_i^{\,T}$$

column vector

right eigenvector of A^T

Instructions for solving