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11:40 AM

$(\mathbf{U}(t) = \mathbf{P}\mathbf{X}(t))$.

Modal analysis $P = [\phi_1 | \phi_2 | \phi_3 | \dots | \phi_n] = \Phi$

$K\phi_i = \omega_i^2 M \phi_i$ → natural modes

ω_i → natural frequencies

$P \to \Phi$

$U = \Phi X$

$$\begin{cases} \phi_i^T K \phi_j = 0 \\ \phi_i^T M \phi_j = 0 \end{cases} \quad (i \neq j)$$

M-orthonormal property $\phi_i^T M \phi_i = 1$ $\quad \phi_i^T K \phi_i = \omega_i^2$

Φ^{-1}

$$\left.\begin{aligned} \Phi^T M \Phi &= I_{n\times n} \\ U &= \Phi X \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} X &= \Phi^{-1} U \\ \Phi^{-1} &= \Phi^T M \end{aligned}\right\} \Rightarrow$$

$$\boxed{X = \Phi^T M U}$$ useful in transferring U system

$$\boxed{X = \phi^T M U}$$ Useful in transferring U system

initial condition to X (Modal system) ICs

$$\ddot{X} + \Omega^2 X = \tilde{R} \qquad \tilde{R} = \phi^T R$$

$$\left\{ \begin{array}{l} \ddot{\mathbf{X}}(t) + \boldsymbol{\Omega}^{\mathrm{T}}\mathbf{C}\boldsymbol{\Omega}\dot{\mathbf{X}}(t) + \boldsymbol{\Omega}^2\mathbf{X}(t) = \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{R}(t) \\ \mathbf{X}^0 = \mathbf{X}(t=0) = \underbrace{\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}}_{\phi^{-1}}\mathbf{U}^0 \qquad \dot{\mathbf{X}}^0 = \dot{\mathbf{X}}(t=0) = \underbrace{\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}}_{\phi^{-1}}\dot{\mathbf{U}}^0 \end{array} \right.$$

- There are three important ranges of $\hat{\omega}$ that we observe from this equation
  1. $\hat{\omega} \ll \omega$ very slow varying load: $D \approx 1$: That is, we are in quasi-static regime and ignoring inertia effects $\ddot{x}$ (and damping as we discuss later $\dot{x}$) is reasonable. Basically, loading rate is so slow that with any increment of loading the system has enough time to reach to a static equilibrium which is why we can ignore $\ddot{x}$ (and $\dot{x}$). In fact, for quasi-static loading regime, we can solve the solution by ignoring $\mathbf{M}$ (and $\mathbf{C}$) in (174) and have $\mathbf{K}\Delta\mathbf{U} = \Delta\mathbf{R}$ between time steps.
  2. $\hat{\omega} \approx \omega$ which is at or near resonance: We have the largest D. For an undamped oscillator $D \to \infty$ as $\hat{\omega} \to \omega$, *i.e.*, when the loading resonance occurs. Later, we show that D remains bounded when damping is added. Still D can get larger than unity for $\hat{\omega}$ near the undamped resonance frequency.
  3. $\hat{\omega} \gg \omega$ very fast varying/oscillating load: In this case the load oscillates so fast that the SDOF system does not have time to respond and basically dynamic respond would be close to zero. That is $D \to 0$ when $\hat{\omega}/\omega \to \infty$.

$\dfrac{\hat{\omega}}{\omega}$ → $= 0$ quasistatic $D = 1$

$= O(1)$ D very large

$\to \infty$ $D \to 0$

before

ω fixed $\qquad \hat{\omega}$ varying

$$M\ddot{U} + KU = 1 \quad \sin(\hat{\omega} t)$$

$= R$

Assume

$\omega_1, \ldots, \omega_n$ $\qquad \omega_i$'s are chang

$\hat{\omega}$ fixed

$\hat{\omega}$ $\qquad \ddot{X}_i + \omega_i^2 x_i = R_i \sin(\hat{\omega} t)$

$$\Omega_i = \frac{\hat{\omega}}{\omega_i}$$

$\hat{\omega}$ fixed

$\omega_i \rightarrow$ varying for $i = 1, \ldots, n$

$$\Omega_i = \frac{\hat{\omega}}{\omega_i} \longrightarrow \frac{\hat{\omega}}{\omega_i} = \Omega_i \ll 1$$

quasistatic response $\qquad \omega_i \gg \hat{\omega}$

$$\longrightarrow = O(1)$$

near resonanse $\qquad \omega_i \approx \hat{\omega}$

$\rightarrow 0$ $\quad D=0$ $\quad \omega_i \ll \hat{\omega}$

$X_i = 0$

We can ignore these

we can ignore

too low frequency

quasi static response

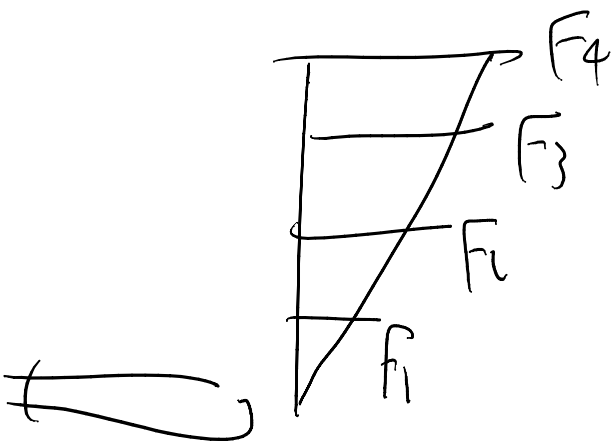
$X_i \approx 0$

can ignore these modes

$D=1$

$D \approx 0$

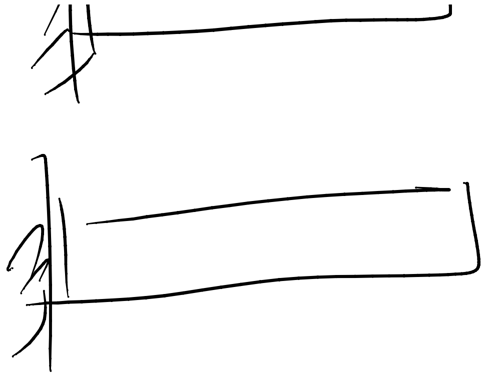
$$\underbrace{M\ddot{U} + C\dot{U}}_{\text{can ignore dynamic part}} + KU = F(t)$$



$$\underbrace{KU(t) = F(t)}$$

$KU = F_0$

$\rightarrow F_1 = 1$ $\quad$ Solution for $t=0$



$\rightarrow F_2 = 2$

solve another static problem

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loading frequency range

ω_1 | ω_2 | ω_3 | ω_4 | ω_5 | ω_6

very low frequencies

p_{min} often $= 0$

medium frequencies

p_{max}

very high frequencies

$D = 0$

D fully dynamic

$D = 1$

Mode i

$$\begin{cases} \ddot{X}_i + 2\omega_i \zeta_i \dot{X}_i + \omega_i^2 X_i = f_i \\ X_i = \phi_i^T M U(0) \\ \dot{X}_i = \phi_i^T M \dot{U}(0) \end{cases}$$

$$\underset{n\times 1}{U(t)} = \sum_{i=1}^{n} \underset{n\times 1}{\phi_i} \, X_i(t)$$

mode shape i

to mathematically solve this system

$$\boxed{M\ddot{U} + C\dot{U} + KU = R}$$

Approximate this as

these are the modes around actual load frequency content

$$* \quad U(t) = \sum_{i=P_{min}}^{P_{max}} X_i(t)\, \phi_i$$

- generally $P_{min} = 1$

- How to we take care of modes $i > P_{max}$ which are not added to the solution

How do we add quasi static contribution

of higher modes ?

$$K \overbrace{\Delta U}^{\text{from higher modes}} = \underbrace{\Delta R}_{\text{error in the loading because we only consider modes } P_{min}(1) \to P_{max}}$$

$$U_{dynamic}(t) = \sum_{P_{min} \simeq 1}^{P} \phi_i X_i(t)$$

small time step for stability

$$\Delta U(t)_{quasistatic} = K^{-1} \underbrace{\Delta R(t)}_{\text{load error}} \longrightarrow \text{time step based on accuracy}$$

Total $U = U_{dynamic} + \Delta U_{quasistatic}$

$\Delta R = ?$

- To add the quasistatic contribution of loading through the higher modes $(1 > p)$ that we did need include in the modal analysis (199) we compute the error in the load vector. Since $\mathbf{R} = \sum_{i=1}^{n} r_i \mathbf{M}\boldsymbol{\Phi}_i$ the error in $\mathbf{R}$ is,

$$\Delta \mathbf{R} = \mathbf{R} - \sum_{i=1}^{p} r_i \mathbf{M}\boldsymbol{\Phi}_i \tag{200}$$

Modal analysis vs. Direct numerical integration of $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$ 226

The choice between computing natural frequencies / modes vs. direct temporal integration of $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$ depends on various aspects.

- **Need for natural frequencies / modes**: In Many applications, regardless of the need to solve $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$, we need to obtain natural frequencies and modes which warrants a modal analysis.
- **Load frequency band** The frequency band of the loadings (BCs, ICs, body force) to a large extend determine how many modes (p) should be included in a modal analysis. We can define two classes of problems:
  1. Structural dynamic problems: Only the first few terms are sufficient for an accurate solution with modal analysis. For example, for earthquake loading in some cases only the 10 lowest modes need to be considered [Bathe, 2006]. If instead of using modal analysis, we directly want to integrate $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$ in time, an implicit scheme is preferred because from accuracy perspective large time steps can be taken without affecting the solution much. Thus, the very small time step restriction of explicit methods can render them inefficient.
  2. Wave propagation problems: The loading frequency is very broadband. For example, in blast of shock loading p can be as high as $2/3n$ [Bathe, 2006]. Often, for wave propagation problems explicit numerical integration schemes are used because they are inexpensive and their restrictive time step is not of major concern because from accuracy perspective small time steps should be taken.

Modal analysis vs. Direct numerical integration of $\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$ 227

Note: For certain vibration problems where loading has a narrow frequency band but the content is high frequency, *i.e.*, that is both $\hat{\omega}_m$ $\hat{\omega}_M$ are high but close to each other, we can omit the lowest natural modes whose frequencies are much smaller than $\hat{\omega}_m$ in the analysis. This reduced the number of modes that need to be considered.

- **Linearity of the problem**: Modal analysis is restricted to linear problems. Although, there may be cases that the nonlinear response can be linearized about the current state or approaches that can expand the applicability of such eigen mode analyses.
- **Influence of damping term**: If the damping term is nonzero AND nondiagonalizable with modal analysis we cannot directly use modal analysis for the solution of (174). Although, under structural dynamic loading we still can consider a much fewer modes $p \ll n$ but in this case p xi terms will be coupled through the damping terms in their corresponding temporal ODEs. For further discussion refer to [Bathe, 2006] Example 9.11.

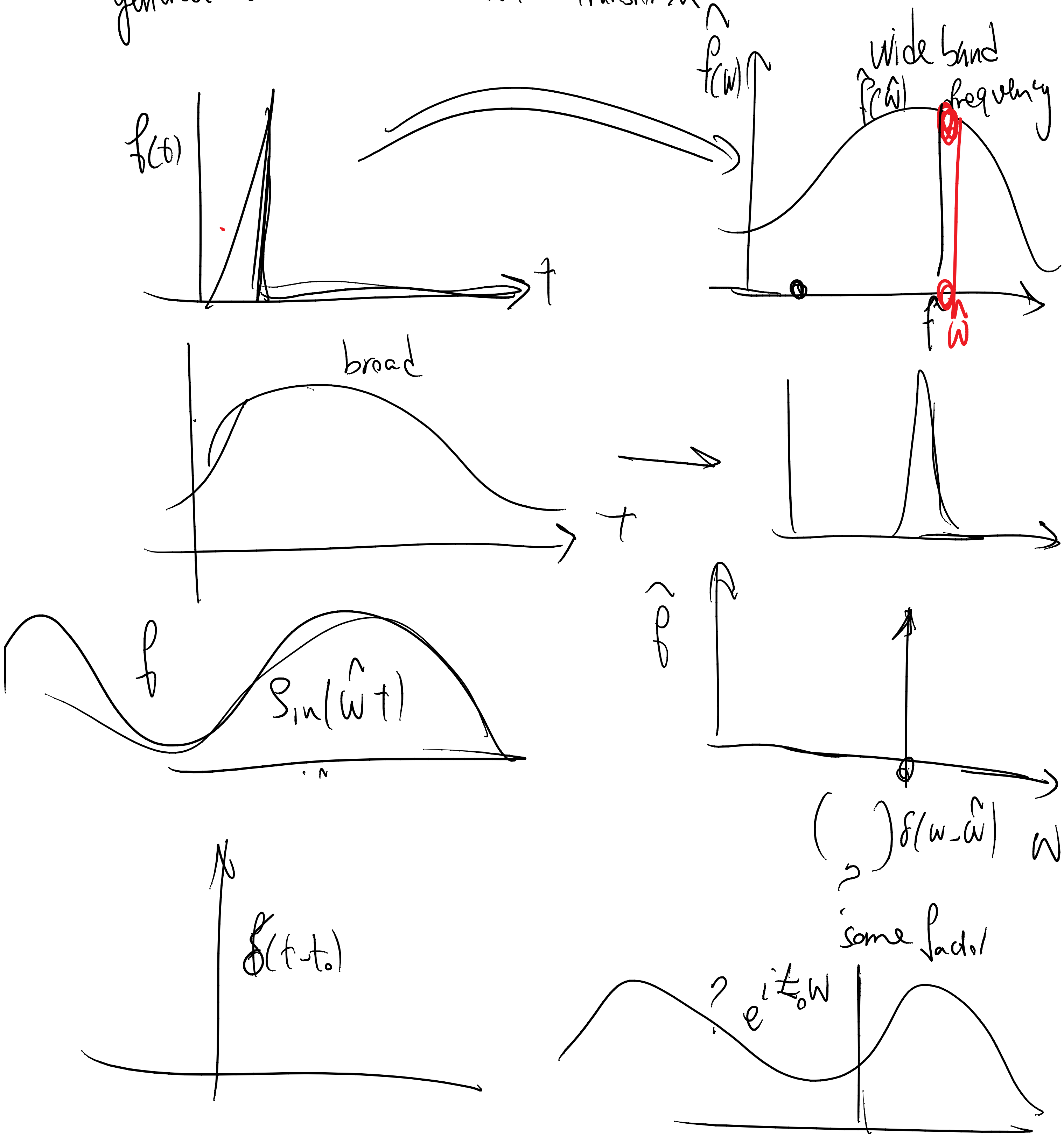
$j^2 = -1$

$$\overline{f}(\hat{\omega}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-j\hat{\omega}t}\, \mathrm{d}t \quad \Leftrightarrow \tag{203a}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f}(\hat{\omega}) e^{j\hat{\omega}t} \mathrm{d}\hat{\omega} \tag{203b}$$

general behavior of Fourier Transform

general behavior of Fourier Transform



Solving a SDOF with damping:

$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f(t)$

Take the Fourier transform $\quad x(t) \rightarrow \bar{x}(\hat{\omega})$

$$\overline{\ddot{x}}(t) = (j\hat{\omega})^2 \, \bar{x}(t)$$

$$(j\hat{\omega})^2 \bar{x} + 2\xi\omega\hat{\omega} j \bar{x} + \omega^2 x = \bar{f}(\hat{\omega})$$

$$\left[\left(-\hat{\omega}^2 + \omega^2\right) + 2\xi\omega\hat{\omega} j\right] \bar{x} = \bar{f}(\hat{\omega})$$

$$\boxed{\bar{x}_{dyn} = \frac{\bar{f}(\hat{\omega})}{(\omega^2 - \hat{\omega}^2) + 2\xi\omega\hat{\omega} j}}$$

if we had static solution

$$\cancel{\ddot{x}} \cancel{+ 2\xi\omega\dot{x}} + \omega^2 x = f$$

$$\omega^2 \bar{x}_{q.stat}(\hat{\omega}) = \bar{f}(\hat{\omega})$$

$$\Omega = \frac{x_{dynamic}}{\quad} =$$

$\Omega = \dfrac{x_{dynamic}}{x_{static}} =$

Damping in a SDOF problem

- Recalling that $\Omega = \frac{\hat{\omega}}{\omega}$ we define ratio of dynamic to static solution,

$$H(\Omega, \xi) := \frac{\overline{x}_{\text{dyn}}(\hat{\omega})}{\overline{x}_{\text{stat}}(\hat{\omega})} = \frac{1}{(1-\Omega^2) + 2j\xi\Omega}, \qquad \Omega = \frac{\hat{\omega}}{\omega}$$

$\dfrac{a+bi}{\sqrt{a^2+b^2}}$