2016/03/07

Monday, March 07, 2016
11:34 AM

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f(t)$$

$$\overline{x}_{\text{dyn}}(\hat{\omega}) = \frac{\overline{x}(\hat{\omega})}{(\omega^2 - \hat{\omega}^2) + 2j\xi\omega\hat{\omega}}$$

$$\overline{x}_{\text{stat}}(\hat{\omega}) = \frac{\overline{f}(\hat{\omega})}{\omega^2}$$

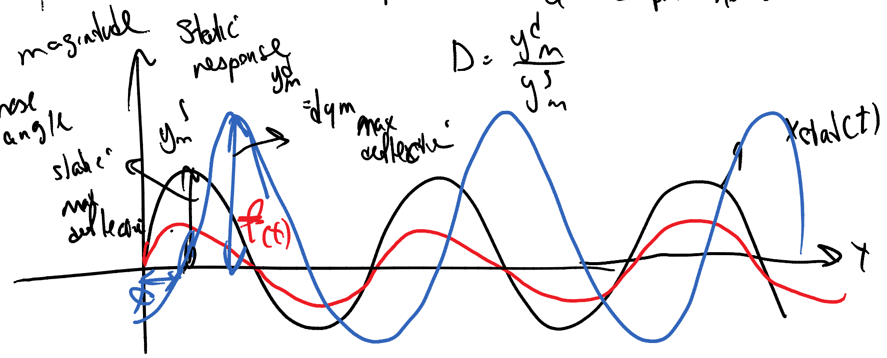
$$H(\Omega, \xi) := \frac{\overline{x}_{\text{dyn}}(\hat{\omega})}{\overline{x}_{\text{stat}}(\hat{\omega})} = \boxed{\frac{1}{(1-\Omega^2) + 2j\xi\Omega}}, \quad \Omega = \frac{\hat{\omega}}{\omega}$$

ratio of dynamic to static response is a complex number

$r e^{i\phi}$ → phase angle

r → magnitude

$D = \frac{y^d_m}{y^s_m}$



$|H| = r = D$

dynamic amplification factor

$$\boxed{D(\Omega, \xi) = |H(\Omega, \xi)| = \frac{1}{\sqrt{(1-\Omega^2)^2 + (2\xi\Omega)^2}}}$$

$D \to \infty$ at resonance when $\xi \neq 0$

$\Omega = \frac{\hat{\omega}}{\omega}$ → load frequency, → natural frequency

$$D_M|_\Omega(\Omega, \xi) = D(\Omega_M, \xi) = \begin{cases} \frac{1}{2\xi\sqrt{1-\xi^2}} & \xi \le \frac{\sqrt{2}}{2} \quad (\Omega_M = \sqrt{1-2\xi^2}) \\ 1 & \text{otherwise} \quad (\Omega_M = 0;\ i.e., \text{static loading}) \end{cases}$$

→ where maximum response is $\Omega_M \neq 1$

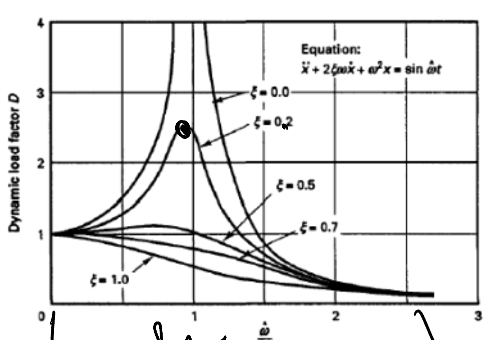


Figure 9.3 The dynamic load factor

$D \to 0$

static response

dynamic

Even with damping we can solve a SDOF exactly with Duhamel's integral

- Furthermore the dynamic solution to (202) ($\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f(t)$) is obtained by the Duhamel integral:

$$x(t) = \frac{1}{\tilde{\omega}} \int_0^t f(\tau) e^{-\xi\omega(t-\tau)} \sin\tilde{\omega}(t-\tau) d\tau + e^{-\xi\omega t}(\alpha \sin\tilde{\omega}t + \beta\cos\tilde{\omega}t), \qquad \text{where} \quad \tilde{\omega} := \omega\sqrt{1-\xi^2}$$

We rarely use this formulate if we cannot come of with a closed form expression

$$\ddot{x}_i + 2\xi_i \omega_i \dot{x}_i + \omega_i^2 x_i = f_i(t)$$

ξ_i: i'th damping factor

ω_i: i'th natural frequency

once we have all $x_i(t)$

$$U_i(t) = \sum_{i=1}^{\cancel{n}} x_i(t)\, \phi_i$$

mode shape ϕ_i

replace this with p in structural dynamic applications $p \ll n$

How to deal with Damping

$$M\ddot{U} + C\dot{U} + KU = R$$

$$\boxed{\ddot{X} + \phi^T C \phi \dot{X} + \Omega^2 X = \tilde{R}}$$

$$\Omega = \begin{bmatrix} \omega_1 & & \\ & \ddots & \\ & & \omega_n \end{bmatrix} \qquad \tilde{R} = \phi^T M R$$

if $C = 0$

the system is decoupled. How about when $C \neq 0$

$\tilde{C} = \phi^T C \phi$

$$\ddot{x}_1 + \omega_1^2 x_1 + \tilde{C}_{11}\dot{x}_1 + \underbrace{\tilde{C}_{12}\dot{x}_2 + \cdots + \tilde{C}_{1n}\dot{x}_n}_{\text{the equations are NOT decoupled in general}} = \tilde{f}_1(t)$$

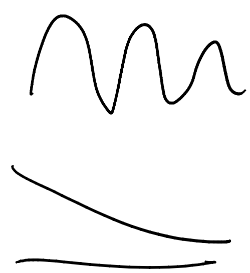
— In general we do not even assemble a C matrix

$M\ddot{U} + KU = R$

$$\ddot{x}_i + \boxed{2\xi_i \omega_i \dot{x}_i} + \omega_i^2 x_i = f_i$$

we add this term

$\xi_i \rightarrow$ Nondimensional damping coeff. $\begin{cases} <1 & \text{underdamped} \\ =1 & \text{critically damped} \\ >1 & \text{overdamped} \end{cases}$



Assume we have $\xi_1, \xi_2, \ldots, \xi_n$

: damping coefficients for mode 1 … mode n

$$\phi^T C \phi = \begin{bmatrix} 2\omega_1\xi_1 & & \\ & \ddots & \\ & & 2\omega_n\xi_n \end{bmatrix}$$

$\Rightarrow$ C must have been $\phi^{-T} \begin{bmatrix} 2\omega_1\xi_1 & & \\ & \ddots & \\ & & 2\omega_n\xi_n \end{bmatrix} \phi^{-1}$

$\phi^{-1} = \phi^T M$

How do we measure ξ_i experimentally? vibration

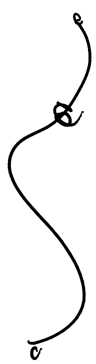
ω_i , ϕ_i $\qquad \ddot{x}_i + 2\xi_i\,\omega_i\,\dot{x}_i + \omega_i^2 x_i = 0$

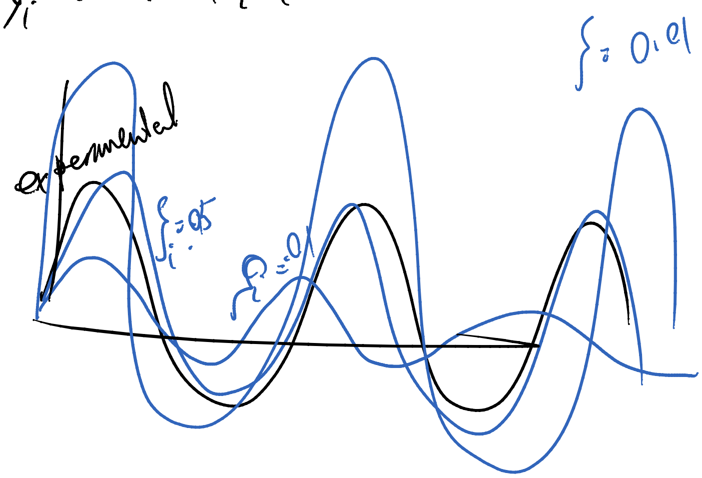
$$U = \sum_{i=1}^{n} x_i(t)\,\phi_i$$

If one mode is activated

$$U = \phi_i\, x_i(t)$$

$$\ddot{x}_i + 2\overset{?}{\xi_i}\,\omega_i\,\dot{x}_i + \omega_i^2 x_i = 0$$





In modal analysis we do not need C.
But in some cases we need C. When?
If we do direct numerical integration of $\quad M\ddot{U} + C\dot{U} + KU = R$

we need C

- If for some reason, the explicit form of $\mathbf{C}$ is required, *e.g.*, when (174) ($\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$) is numerically integrated in time by explicit or implicit methods, we can form $\mathbf{C}$ by Caughey series,

$$\mathbf{C} = \mathbf{M}\sum_{k=0}^{r-1} a_k[\mathbf{M}^{-1}\mathbf{K}]^k, \qquad \text{where } a_k \text{ are solved from } r \text{ simultaneous equations :} \tag{212a}$$

$$\xi_i = \frac{1}{2}\left(\frac{a_0}{\omega_i} + a_1\omega_i + a_2\omega_i^3 + \cdots + a_{r-1}\omega_i^{2r-3}\right), \qquad i = 0, \cdots, (r-1) \tag{212b}$$

and r is the number of damping coefficients given to define $\mathbf{C}$.

If we only consider two terms

- For $r = 2$ we recover,

If we only consider two terms

- For $r = 2$ we recover,

given from experiments $\mathbf{C} = a_0\mathbf{M} + a_1\mathbf{K}$ form Rayleigh Damping matrix

This is called Rayleigh damping matrix

$$\left.\begin{array}{ll}\xi_0 & = \frac{1}{2}\left(\frac{a_0}{\omega_0} + a_1\omega_0\right) \\ \xi_1 & = \frac{1}{2}\left(\frac{a_0}{\omega_1} + a_1\omega_1\right)\end{array}\right\} \Rightarrow \left\{\begin{array}{ll} a_0 & = 2\omega_0\omega_0\frac{\xi_0\omega_1 - \xi_1\omega_0}{\omega_1^2 - \omega_0^2} \\ a_1 & = 2\frac{\xi_1\omega_1 - \xi_0\omega_0}{\omega_1^2 - \omega_0^2}\end{array}\right.$$

(ϕ_0^T) (ϕ_0)

$$C = \underbrace{a_0 M + a_1 K}_{\text{do not correspond to mode I \& II}}$$

$$\underbrace{\phi_0^T C \phi_0}_{} = a_0 \underbrace{\phi_0^T M \phi_0}_{} + a_1 \underbrace{\phi_0^T K \phi_0}_{}$$

$$2\xi_0 \omega_0 = a_0 \, 1 + a_1 \omega_0^2$$

$\phi^T M \phi = 1$

$\phi^T K \phi = \Omega^2$

$$\begin{bmatrix} \omega_1^2 & & \\ & \ddots & \\ & & \omega_n^2 \end{bmatrix}$$

ξ_i $i > 1$ (no summation on i)

$$2\xi_i \omega_i = a_0 + a_1 \omega_i^2$$

$$\Rightarrow \boxed{\xi_i = \frac{a_0 + a_1\omega_i^2}{2\omega_i}}$$

if we only use ξ_0 & ξ_1 to form

$$C = a_0 M + a_1 K$$

Other damping coefficients must be

$$\boxed{\xi_i = \frac{a_0 + a_1\omega_i^2}{2\omega_i}}$$

Mode 3 & higher $(i \geq 2)$

$$\ddot{x}_i + 2\xi_i \omega_i \dot{x}_i + \omega_i^2 x_i = f_i$$

no summation i

$\downarrow$ mod 3 & higher obtained from

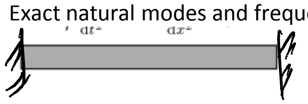
Side note

$$M\ddot{U} + C\dot{U} + KU = 0$$

$$U = \phi e^{i\omega t}$$ $\hookrightarrow$ a complex value

we can still get modal shapes & frequencies in complex plane

Exact natural modes and frequencies

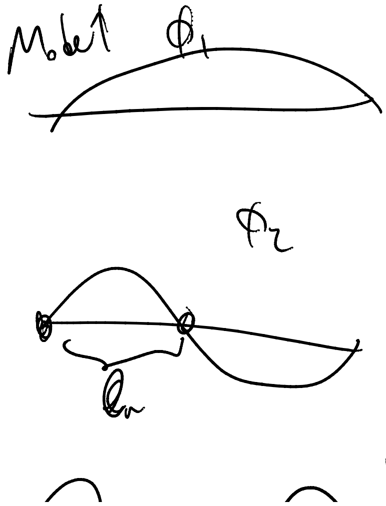


$\omega_n = n\pi \frac{c}{L}$

$$\phi_n = \sin\left(\frac{\omega_n x}{c}\right) = \sin\left(\frac{n\pi x}{L}\right)$$

1D $\quad \omega_n \propto n$ at high n

Mode 1 ϕ_1



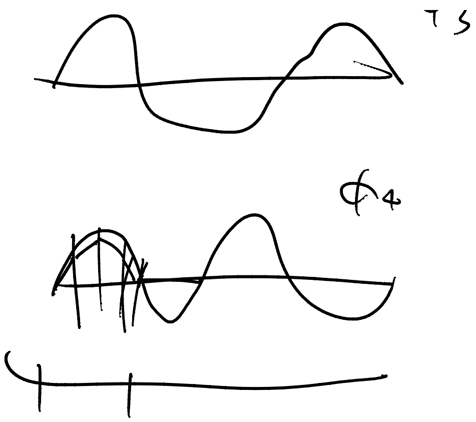
ϕ_2

L

l_w

$$l_w \propto \frac{L}{n}$$

ϕ_3



need few elements to capture mode 4 accurately
linear

Error analysis for natural modes and frequencies
Preliminaries

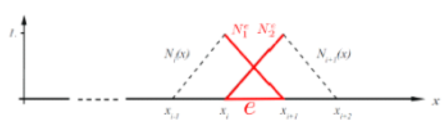
Preliminaries: FEM global continuity level $m-1$

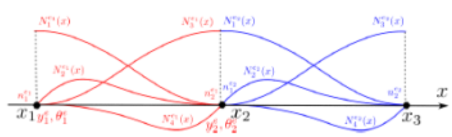
3.1.7 Error analysis for natural frequencies and natural modes

- If the differential equation has $2m$ highest spatial derivative, shape functions must be globally $\mathcal{C}^{m-1}$ continuous.
- Below, two cases for bar and beam examples are shown:

-	Bar	Beam
PDE	$\rho A \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - EA\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0$	$\rho A \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - EI\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = 0$
$2m$	2	4
Global continuity $\mathcal{C}^{m-1}$	0	1
m	1	2

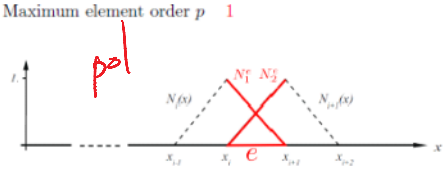
Preliminaries: FEM global continuity level $m-1$

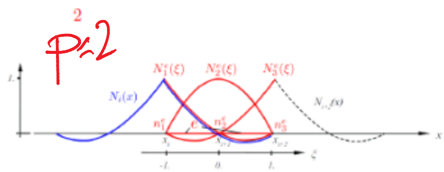




Preliminaries: FEM polynomial order p

-	Bar (1storder)	Bar (2nd)
Sample shape function	$N_1 = 1 - \frac{x^1}{L}$	$N_1 = 1 - \frac{3}{2}x + \frac{1}{2}x^2$
Maximum element order p	1	2





- Note that the element maximum polynomial order p is not the same as minimum global required continuity $m-1$.
- For example, in the figure both elements are for the bar element with $m-1=0$ (C^0 global continuity).
- Yet, the element on the left is 0th order ($p=0$) and on the right 1storder ($p=0$).

Errors for natural modes and natural frequencies

A priori error estimates for natural frequencies and modes

- *A priori* error estimates for natural frequencies and natural modes are in the form,

numerical natural mode i

$$0 \le \omega_i^h - \omega_i \le C h^{2(p+1-m)} \omega_i^{\frac{2p+2-m}{m}} \tag{222a}$$

$$||\Phi_i^h - \Phi_i||_m \le C h^{(p+1-m)} \omega_i^{\frac{p+1}{m}} \tag{222b}$$

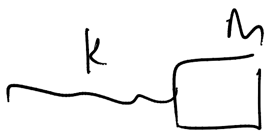
exa ral f ency i

What does this mean?

$$0 \le \omega_i^h - \omega_i$$

$\omega_i < \omega_i^h$

numerical frequencies are higher than the exact one



$\omega = \sqrt{\frac{k}{m}}$ $\quad k \uparrow \ \omega \uparrow$

Finite Element Solutions are stiffer in general

grid resolution h = the largest element size (size of an element is the radius of its circumscribing circle (2D) / sphere (3D))

1. $0 \le \omega_i^h - \omega_i$, *i.e.*, having $\omega_i^h \le \omega_i$ is not preserved once the Galerkin rules are violated [?] (e.g., when reduced integration or incompatible modes are employed or when lumped mass matrix is used).
2. The rate of convergence (*i.e.*, power of h) of eigenvalues is twice that of eigenfunctions in the H^m (Hilbert m norm) [compare (222a) and (222b)]. That is,

   Natural frequencies converge twice faster than natural modes

3. The appearance of powers of the natural frequencies on the right-hand sides of (222a) $\omega_i^{\frac{2p+2-m}{m}}$ and (222b) $\omega_i^{\frac{p+1}{m}}$ suggests that the **quality of approximation deteriorates for higher modes**. Recall that $\omega_0 < \omega_1 < \cdots < \omega_n$. This can be explained that higher modes have higher spatial variability (wave number) and for the same resolution of FEM mesh h it is more difficult to capture the exact solution.

4. **K**, **M** (and **C**) are often integrated numerically, *i.e.*, by quadrature.

(a) For the convergence rates in h in (222) to hold:

The quadrature rule must be accurate enough to exactly integrate all monomials through order $\bar{p} + p - 2m$ where

$$\bar{p} = \text{Order of the highest-order monomial appearing in the element shape functions,}$$
$$p = \text{Order of the element}$$
$$m - 1 = \text{Level of global continuity of FEM shape functions}$$

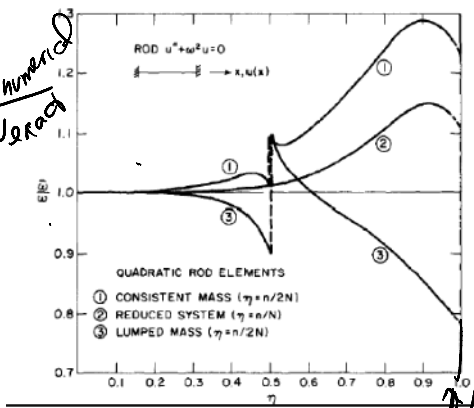
(b) A sufficient condition for the convergence of modal quantities (as $h \to 0$) is
The quadrature rule must be accurate enough to exactly integrate all monomials through order $\bar{p} - m$ (a weaker condition that having the full convergence rates)

natural mode # n

$\eta = \frac{n}{N}$

the number of modes an FEM can capure

$\frac{W_{numerical}}{W_{exact}}$



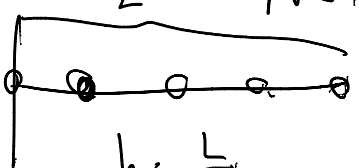
$\frac{N}{N}$

Natural frequency spectra

$n \propto \frac{1}{N}$

N N=10

L N=4



$h = \frac{L}{N}$

$h \propto \frac{1}{N}$

N is high

$h_{\text{oscillations for mode } n} \propto \frac{1}{n}$

$\frac{h_{FEM}}{h_{mode_n}} = \frac{n}{N}$