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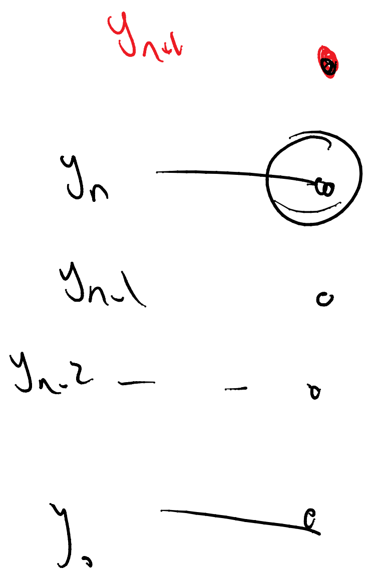
4.3 Linear Multistep (LMS) methods

- Consider the first order ODE,

$$\begin{aligned}\dot{\mathbf{y}} &= f(\mathbf{y}, t) \\ &= \mathbf{G}\mathbf{y} + \mathbf{H}(t)\end{aligned}$$

$\dot y = \underbrace{f(y,t)}$

possibly nonlinear function in y



$$\dot y_n \approx \frac{y_{n+1} - y_n}{\Delta t}$$

forward Euler $y_{n+1} = y_n + \Delta t\, f(y_n, t)$

$$\dot y_{n+1} = \frac{y_{n+1} - y_n}{\Delta t}$$

$$\dot y_{n+1} = f(y_{n+1}, t)$$

$$\Rightarrow \boxed{y_{n+1} - \Delta t\, f(y_{n+1}, t)} = y_n$$

implicit because y_{n+1} appears implicitly in this equation.

$O(\Delta t)$

accurate FD schemes

More accurate difference ?

$$\dot y_n \approx \frac{y_{n+1} - y_{n-1}}{2\Delta t} + O(\Delta t^3)$$

$$y_n \approx \frac{\cdots - y_{n-1}}{2\Delta t} + O(\Delta t^2)$$

FD method will be $O(\Delta t^2)$ accurate

$$\dot{y}_n = f(y_n, t_n)$$

$$\to \quad y_{n+1} = 2\Delta t\, f(y_n, t_n) + y_{n-1}$$

y_{n+1} o

y_n o
y_{n-1} o } 2-step method

What is the advantage of having more steps back?

Higher order accuracy?

Examples of LMS methods (in this case applied to temporally second order FEM elastodynamic discretization)

$$\mathbf{M\ddot{U}} + \mathbf{C\dot{U}} + \mathbf{KU} = \mathbf{R}$$

time step $n-1$ time step n time step $n+1$

$$^t\ddot{\mathbf{U}} = \frac{1}{\Delta t^2}\left({}^{t-\Delta t}\mathbf{U} - 2\,{}^t\mathbf{U} + {}^{t+\Delta t}\mathbf{U}\right)$$

$$^t\dot{\mathbf{U}} = \frac{1}{2\Delta t}\left(-{}^{t-\Delta t}\mathbf{U} + {}^{t+\Delta t}\mathbf{U}\right)$$

good

$$M\,{}^t\ddot{U} + C\,{}^t\dot{U} + K\,{}^tU = {}^tR$$

$t \to t+\Delta t$

$t+\Delta t$ o, $n+1$ time step

$\cdot\,+1$ $\cdot\,+1$

$\dot{u}$ (o) $\ddot{u}$ (o) -2

t   o   n   time step

o -1 o $+1$

$t-\Delta t$ o $n-1$ n

2 - Step $t, t-\Delta t \rightarrow t+\Delta t$

explicit   because the equation is written for t (current time step)

Update equation

1 $$\boxed{\left(\frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{C}\right)} {}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{R} - \left(\mathbf{K} - \frac{2}{\Delta t^2}\mathbf{M}\right){}^{t}\mathbf{U} - \left(\frac{1}{\Delta t^2}\mathbf{M} - \frac{1}{2\Delta t}\mathbf{C}\right){}^{t-\Delta t}\mathbf{U}$$

As with most explicit methods (K stiffness does not appear on the LHS)

$$\hat{\mathrm{M}} = \frac{1}{\Delta t^2}\mathrm{M} + \frac{1}{2\Delta t}\mathrm{C}, \quad \text{where} \quad \hat{\mathrm{M}}\mathrm{U}^{n+1} = \mathrm{R}^n$$

If C = 0

$$\hat{\mathrm{M}} = \frac{1}{\Delta t^2}\mathrm{M}.$$

Now if we use a lumped mass matrix the equation becomes trivial

$$\underbrace{{}^{t+\Delta t}U_i}_{} = {}^{t}\hat{R}_i\left(\frac{\Delta t^2}{m_{ii}}\right) \quad \text{for} \quad {}^{t}\hat{\mathbf{R}} = {}^{t}\mathbf{R} - \left(\mathbf{K} - \frac{2}{\Delta t^2}\mathbf{M}\right){}^{t}\mathbf{U} - \left(\frac{1}{\Delta t^2}\mathbf{M}\right){}^{t-\Delta t}\mathbf{U}$$

Disadvantages of LMS methods:

1. First few steps where time values -1, … may be needed.
2. Very difficult (or practically impossible) to adjust the time step

t_{n+1}

t_n

Δt can not change

t_{n-1}

How can we have methods that go from time step n to time step n + 1 without the need of previous time step values even for second order elastodynamic problem?

Multivariate single-step methods

LMS

x^{n+1}

x^{n}

x^{n-1}

$\longrightarrow$ $\left(x^{n+1}, \dot{x}^{n+1}, \ddot{x}^{n+1}\right)$

$\Uparrow$

$\left(x^{n}, \dot{x}^{n}, \ddot{x}^{n}\right)$

vector of unknowns for time step n

Examples:

4.4.1 The θ-Wilson method

- In θ-Wilson method acceleration is linearly interpolated between time step t_n (t) to $\theta \Delta t$ after that

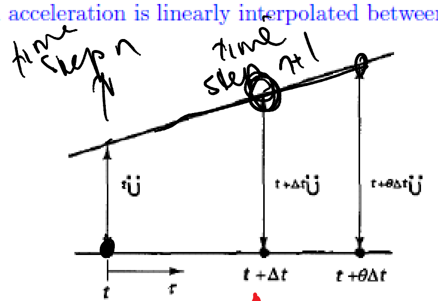


Figure 9.1 Linear acceleration assumption of Wilson θ method

time value $n + \theta \Delta t$

$\theta > 1.37$ for the method to be stable

$${}^{t+\tau}\ddot{\mathbf{U}} = {}^{t}\ddot{\mathbf{U}} + \frac{\tau}{\theta\,\Delta t}\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}}\right)$$

still unknown

we want to land here for time step n+1

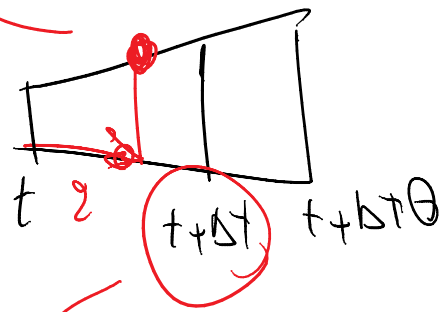
- By twice integration of acceleration equation (253) we obtain equations for $\mathbf{U}$ and $\dot{\mathbf{U}}$:

any $\tau \in [0, \theta\Delta t]$

$$ {}^{t+\tau}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + {}^{t}\ddot{\mathbf{U}}\tau + \frac{\tau^2}{2\theta\,\Delta t}({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}}) $$

$$ {}^{t+\tau}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\tau + \frac{1}{2}{}^{t}\ddot{\mathbf{U}}\tau^2 + \frac{1}{6\theta\,\Delta t}\tau^3({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}}) $$

still unknown



plug $\tau = \Delta t$ to get to next time step values

linear

$$ {}^{t+\Delta t}\dot{U} = g({}^{t+\theta\Delta t}U) $$

Goal: To express everything in terms of displacement of the next time step θ $({}^{t+\Delta t\theta}U)$

$$ {}^{t+\theta\Delta t}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + \frac{\theta\,\Delta t}{2}({}^{t+\theta\Delta t}\ddot{\mathbf{U}} + {}^{t}\ddot{\mathbf{U}}) \quad \text{(a)} $$

$$ {}^{t+\theta\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + \theta\,\Delta t\,{}^{t}\dot{\mathbf{U}} + \frac{\theta^2\,\Delta t^2}{6}({}^{t+\theta\Delta t}\ddot{\mathbf{U}} + 2\,{}^{t}\ddot{\mathbf{U}}) \quad \text{(b)} $$

linear

$$ {}^{t+\theta\Delta t}\ddot{U} = f({}^{t+\Delta\theta}U) $$

By solving (f and g above) we get

- This provides values for the unknowns:

the only unknown

$$ {}^{t+\theta\Delta t}\ddot{\mathbf{U}} = \frac{6}{\theta^2\,\Delta t^2}({}^{t+\theta\Delta t}\mathbf{U} - {}^{t}\mathbf{U}) - \frac{6}{\theta\,\Delta t}{}^{t}\dot{\mathbf{U}} - 2\,{}^{t}\ddot{\mathbf{U}} $$

$$ {}^{t+\theta\Delta t}\dot{\mathbf{U}} = \frac{3}{\theta\,\Delta t}({}^{t+\theta\Delta t}\mathbf{U} - {}^{t}\mathbf{U}) - 2\,{}^{t}\dot{\mathbf{U}} - \frac{\theta\,\Delta t}{2}{}^{t}\ddot{\mathbf{U}} $$

Write the ODE for t + θ Δt

$$ {}^{t+\theta\Delta t}\dot{\mathbf{U}} = \frac{3}{\theta\,\Delta t}\left({}^{t+\theta\Delta t}\mathbf{U} - {}^{t}\mathbf{U}\right) - 2\,{}^{t}\dot{\mathbf{U}} - \frac{\theta\,\Delta t}{2}\,{}^{t}\ddot{\mathbf{U}} $$

Write the ODE for t + $\theta\,\Delta t$

$$ \mathbf{M}\,{}^{t+\theta\Delta t}\ddot{\mathbf{U}} + \mathbf{C}\,{}^{t+\theta\Delta t}\dot{\mathbf{U}} + \mathbf{K}\,{}^{t+\theta\Delta t}\mathbf{U} = {}^{t+\theta\Delta t}\overline{\mathbf{R}} $$

$$ {}^{t+\theta\Delta t}\overline{\mathbf{R}} = {}^{t}\mathbf{R} + \theta\left({}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{R}\right) $$

update equation

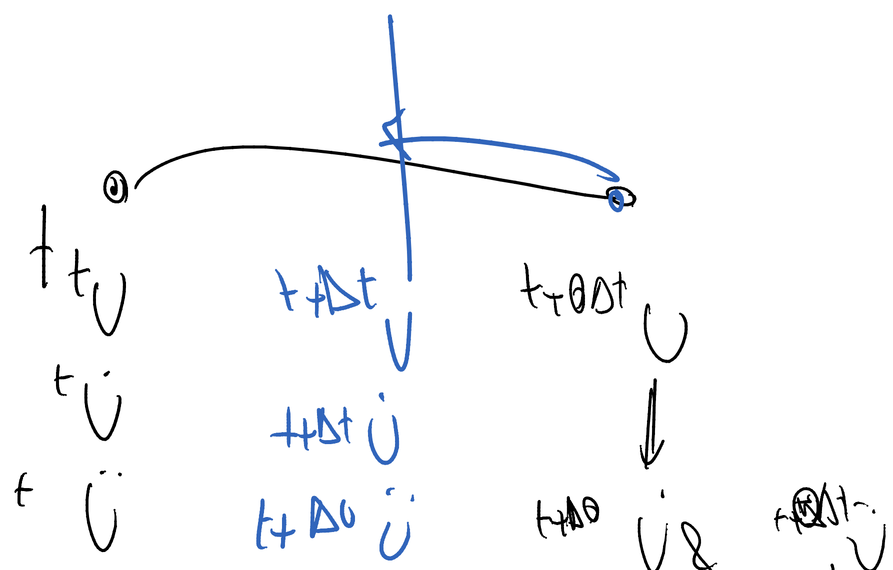
$$ \hat{\mathrm{K}}\,{}^{t+\theta\Delta t}\mathrm{U} = {}^{t+\theta\Delta t}\,\hat{\mathrm{R}} $$

$$ \hat{\mathrm{K}} = \mathrm{K} + \frac{6}{(\theta\Delta t)^2}\mathrm{M} + \frac{3}{\theta\Delta t}\mathrm{C} $$ → effective stiffness matrix

$$ {}^{t+\theta\Delta t}\hat{\mathbf{R}} = {}^{t}\mathbf{R} + \theta\left({}^{t+\Delta t}\mathbf{R} - {}^{t}\mathbf{R}\right) + \mathbf{M}\left(a_0\,{}^{t}\mathbf{U} + a_2\,{}^{t}\dot{\mathbf{U}} + 2\,{}^{t}\ddot{\mathbf{U}}\right) + \mathbf{C}\left(a_1\,{}^{t}\mathbf{U} + 2\,{}^{t}\dot{\mathbf{U}} + a_3\,{}^{t}\ddot{\mathbf{U}}\right) $$

effective force

${}^{t+\theta\Delta t}U$ will be obtained

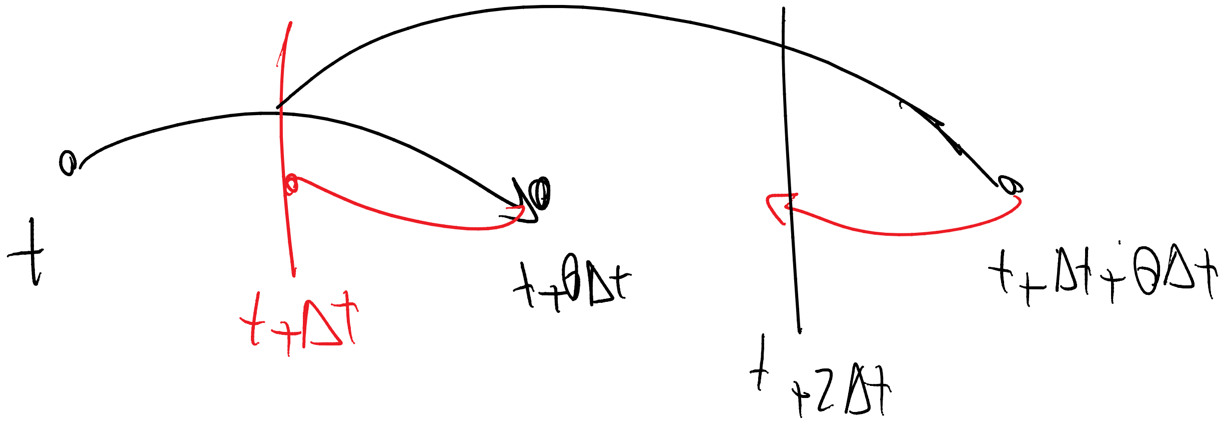


$$ {}^{t+\tau}\dot{\mathbf{U}} = {}^{t}\dot{\mathbf{U}} + {}^{t}\ddot{\mathbf{U}}\tau + \frac{\tau^2}{2\theta\,\Delta t}\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}}\right) $$

$$ {}^{t+\tau}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\tau + \frac{1}{2}\,{}^{t}\ddot{\mathbf{U}}\tau^2 + \frac{1}{6\theta\,\Delta t}\tau^3\left({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^{t}\ddot{\mathbf{U}}\right) $$

plug $\tau = \Delta t$

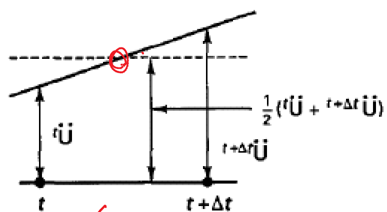
already known



$\theta < 1.37$ unstable

Newmark method

4.4.2 The Newmark method



- In Newmark method, $\mathbf{U}$, $\dot{\mathbf{U}}$ are expressed in terms of $\mathbf{U}, \dot{\mathbf{U}}, \ddot{\mathbf{U}}$ at t_n and ${}^{t+\Delta t}\ddot{\mathbf{U}}$:

$$\boxed{{}^{t+\Delta t}\dot{\mathbf{U}}} = {}^{t}\dot{\mathbf{U}} + [(1-\delta)\,{}^{t}\ddot{\mathbf{U}} + \delta\,\boxed{{}^{t+\Delta t}\ddot{\mathbf{U}}}]\,\Delta t$$
$${}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\,\Delta t + [(\tfrac{1}{2}-\alpha)\,{}^{t}\ddot{\mathbf{U}} + \alpha\,\boxed{{}^{t+\Delta t}\ddot{\mathbf{U}}}]\,\Delta t^2 \quad (259)$$

δ

acceleration

unknown

$t+\Delta t$ $\ddot{U}$

parameters of Newmark method

$$\boxed{{}^{t+\Delta t}\dot{\mathbf{U}}} = {}^{t}\dot{\mathbf{U}} + [(1-\delta)\,{}^{t}\ddot{\mathbf{U}} + \delta\,\boxed{{}^{t+\Delta t}\ddot{\mathbf{U}}}]\,\Delta t \rightarrow t+\Delta t\ \dot{U}$$
$${}^{t+\Delta t}\mathbf{U} = {}^{t}\mathbf{U} + {}^{t}\dot{\mathbf{U}}\,\Delta t + [(\tfrac{1}{2}-\alpha)\,{}^{t}\ddot{\mathbf{U}} + \alpha\,\boxed{{}^{t+\Delta t}\ddot{\mathbf{U}}}]\,\Delta t^2 \rightarrow t+\Delta t\ \ddot{U}$$

in terms of $t+\Delta t$ $\ddot{U}$

in terms of $t+\Delta t$ $\ddot{U}$

$t+\Delta t$ ..

$$M \ddot{U}^{t+\Delta t} + C \dot{U}^{t+\Delta t} + K U^{t+\Delta t} = F^{t+\Delta t}$$

both in terms of $U^{t+\Delta t}$

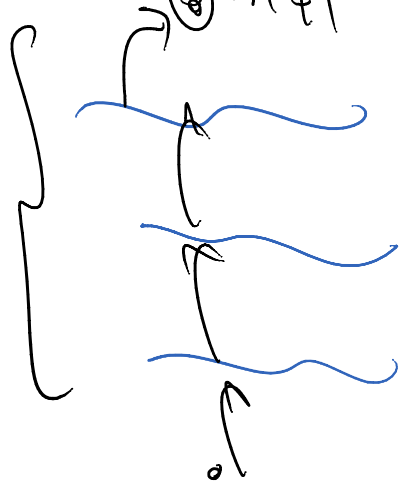
$$\hat{K} U^{t+\Delta t} = \hat{R}$$

look for values in table 9.4

3rd approach
Runge-Kutta method

y_{n+1} ~~& $\dot{y}_{n+1}$~~ @ t_{n+1}

Intermediate values for y & $\dot{y}$



stage 3

stage 2

stage 1

t_n

Not steps

More stages $\Longrightarrow$ higher order accuracy

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) \qquad \text{First order ODE}$$

$$y(t=0) = y_0 \qquad \text{Initial condition (IC)}$$

- Explicit Runge-Kutta (RK) update the solution from time step t_n to t_{n+1} through $s \geq 1$ stages:

weight values

$$y_{n+1} = y_h + \Delta t \sum_{i=1}^{s} d_i k_i \qquad \text{where}$$

slopes

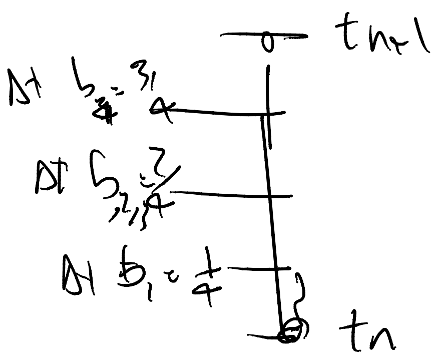
dimension of $\frac{dy}{dt}$: $\dot{y}$

$$k_i = f(t_n + \Delta t b_i, y_n + \Delta t \sum_{j=1}^{i-1} a_{ij} k_j), \qquad 1 \leq i \leq s$$

time factors

$$\dot{y} = f(t, y) = G(t)\, y + N(t)$$

if f is linear



$$\begin{array}{c||ccccc}
c_1 & 0 & \cdots & 0 & 0 \\
c_2 & a_{21} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & 0 & 0 \\
c_s & a_{s1} & \cdots & a_{s,s-1} & 0 \\
\hline\hline
 & b_1 & \cdots & b_{s-1} & b_s
\end{array}$$