2016/03/30

Wednesday, March 30, 2016
11:40 AM

Continue stability analysis of generalized alpha method

$$\begin{cases} \alpha < \frac{1}{2} & \text{Conditionally stable} \quad \Delta t\lambda < \frac{2}{1-2\alpha} \\ \alpha \geq \frac{1}{2} & \text{Unconditionally stable} \end{cases}$$

$\dot{u} + \lambda u = 0$ SDof

$\underset{m\times m}{M}\dot{d} + Kd = 0$ MDof

$\dot{d}$: vector, size m

K: $m \times m$

$\Rightarrow \begin{cases} m \text{ SDof} \\ \dot{d}_{(\ell)} + \lambda_\ell d_\ell = 0 \end{cases}$

stability to d_ℓ requires

$$\begin{cases} \alpha < \frac{1}{2} & \Delta t \leq \frac{1}{\lambda_\ell}\frac{2}{1-2\alpha} \\ \alpha \geq \frac{1}{2} & \text{unconditionally stable} \end{cases}$$

Let's assume $\alpha < \frac{1}{2}$

$\ell = 1 \quad \Delta t \leq \frac{1}{\lambda_1}\frac{2}{1-2\alpha}$

$\ell = 2 \quad \Delta t \leq \frac{1}{\lambda_2}\frac{2}{1-2\alpha}$

$\vdots$

$\ell = m \quad \Delta t \leq \frac{1}{\lambda_m}\frac{2}{1-2\alpha}$

$\Rightarrow$

$l = m \quad \Delta t \leq \frac{1}{\lambda_m} \frac{1}{1-2\alpha}$)

$$\Delta t \leq \text{Min}\left(\frac{1}{\lambda_l}\right) \frac{2}{1-2\alpha}$$

$$\Rightarrow \Delta t \leq \frac{1}{\underbrace{\text{Max}(\lambda_l)}} \frac{2}{1-2\alpha}$$

we need this if $\alpha < \frac{1}{2}$

Maximum frequency (eigenvalue) of

$M\dot{d} + Kd = 0$

$$M\dot{d} + Kd = 0$$

want to directly integrate this without doing the modal analysis to find $\text{Max}(\lambda_l)$

$$M\left(\frac{d_{n+1} - d_n}{\Delta t}\right) + K\left((1-\alpha)d_{n+1} + \alpha d_n\right) = 0$$

$$\left\{\frac{M}{\Delta t} + (1-\alpha)K\right\} d_{n+1} = \left\{\frac{M}{\Delta t} - \alpha K\right\} d_n$$

need to solve

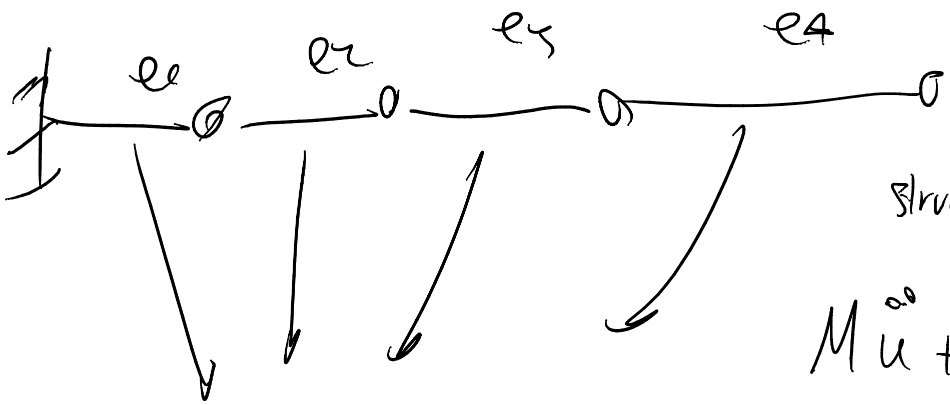
$K\psi_l = \lambda_l M\psi_l$

We don't actually need to do the expensive modal analysis

- Once can prove [Hughes, 2012, Bathe, 2006],

$$\boxed{\lambda_e^m \geq \max_l(\lambda_l^h)} \qquad (311)$$

Maximum frequency of individual elements in the domain



Structural dynamics

$$M\ddot{u} + Ku = 0$$

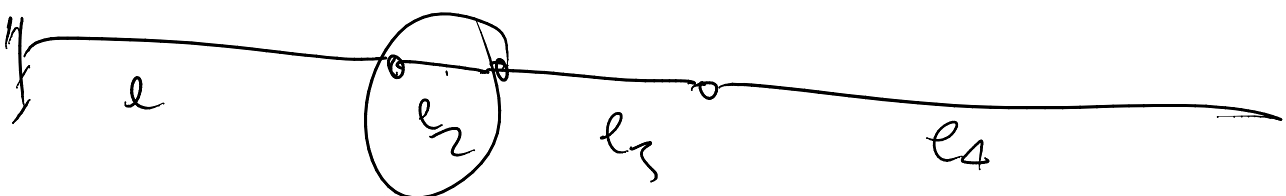
Eigenvalue problem at the element level

$$K\phi = \omega^2 M\phi$$

$$K^e = \frac{AE}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$M_c^e = \frac{A l_e \rho}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

which element has the highest frequency



has the highest frequency

has the highest frequency $\lambda_e^m = \sqrt{12}\ \dfrac{c}{l}$

with constant mass

$$\lambda_e^m > \underbrace{\mathrm{Max}(\lambda_l)}_{\text{max freq. of MDOF}}$$

max freq. of individual elements

expensive to calculate

$$\frac{1}{\lambda_e^m} < \frac{1}{\mathrm{Max}(\lambda_l)}$$

if $\Delta t < \dfrac{1}{\lambda_e^m} \dfrac{2}{1-2\alpha} < \dfrac{1}{\mathrm{Max}(\lambda_l)} \dfrac{2}{1-2\alpha}$

easy to calculate

$$\Rightarrow \boxed{\Delta t < \frac{1}{\mathrm{Max}(\lambda_l)} \frac{2}{1-2\alpha}}$$

We talked about stability

Second concept is consistency

We have the numerical method update equation

$$d_{n+1} = \color{red}{A} d_n + L_n \quad \Rightarrow \quad \boxed{d_{n+1} - Ad_n - L_n = 0}$$

$$L_n = \Delta t \frac{(1-\alpha) f_n + \alpha f_{n+1}}{1 + \alpha \Delta t \lambda}$$

$$\dot{d}^{n+\alpha} + \lambda d^{n+\alpha} = f^{n+\alpha}$$

$$\frac{d_{n+1} - d_n}{\Delta t} \qquad \{(1-\alpha) d_n + \alpha d_{n+1}\}$$

t_{n+1}

$\alpha \Delta t$ $\quad n+\alpha$

t_n

$$\frac{d_{n+1} - d_n}{\Delta t} + \lambda \{(1-\alpha) d_n + \alpha d_{n+1}\} = f^{n+\alpha}$$

$$(1 + \alpha \Delta t \lambda)\, d_{n+1} = (1 - \lambda \Delta t (1-\alpha))\, d_n + \Delta t f_n$$

$$d_{n+1} = \color{red}{A} d_n + \underbrace{\Delta t f^{n+1}}_{L_n}$$

$$A = \frac{1 - (1-\alpha)\lambda \Delta t}{1 + \alpha \Delta t}$$

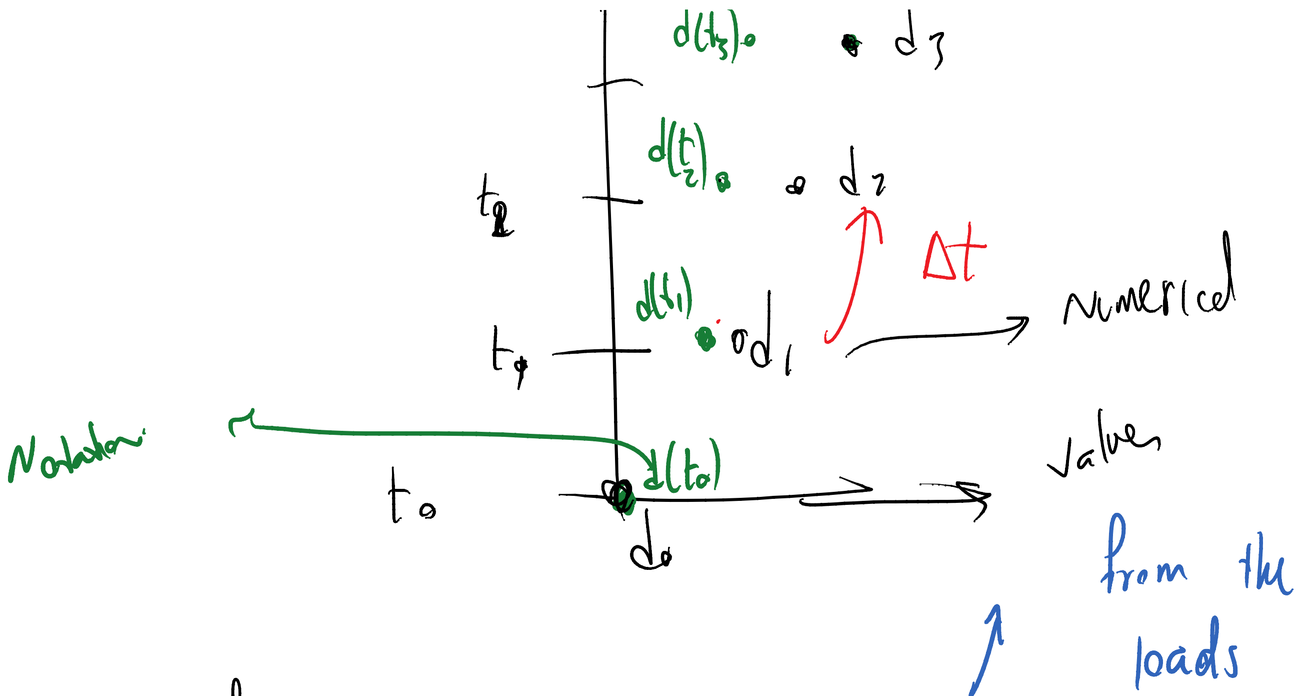
$$\boxed{d_{n+1} = A d_n + L_n}$$

Numerical update

$$A = \frac{1 - (1-\alpha)\lambda \Delta t}{1 + \alpha \Delta t}$$

$$L_n = \frac{f^{n+\alpha}}{1 + \alpha \Delta t} = \frac{(1-\alpha) f_n + \alpha f_{n+1}}{1 + \alpha \Delta t}$$

$d(t_3)$ $\quad d_3$



Numerical $\qquad d_{n+1} = A d_n + L_n$

What is the exact solution update?

$$\dot d + \lambda d = 0 \qquad d(t_{n+1}) = d(t_n)\, e^{-\lambda [t_{n+1} - t_n]}$$

Consistency: is concerned with closeness "consistency" of the numerical method & the underlying $\left\{ \begin{matrix} \text{exact solution} \\ \text{PDE} \end{matrix} \right\}$ at ONE update (LOCAL update)

What is consistency for ODE's in this case

Generalized alpha method.

from load

$$+\left\{\; d_{n+1} = A d_n + L_n \right.$$

Numerical

$$-\left\{\; d(t_{n+1}) = A\, d(t_n) + L_n + \Delta t\, \underbrace{\tau(t_n)}_{\text{truncation error}} \right.$$

$$d_{n+1} - d(t_{n+1}) = A\left\{ d_n - d(t_n) \right\} - \Delta t\, \tau(t_n)$$

$$e_{n+1} = A e_n - \Delta t\, \tau(t_n)$$

— 1 Consistency requires $\tau(t_n) \leq C \Delta t^k$

$k \geq 0$

order of accuracy

— 2 Stability $|A| \leq 1$

Do they result in convergence

$$\left.\begin{aligned} e_{n+1} &= A e_n - \Delta t\, \tau(t_n) \\ e_n &= A e_{n-1} - \Delta t\, \tau(t_{n-1}) \end{aligned}\right\} \Rightarrow$$

$$e_{n+1} = A\left(A e_{n-1} - \Delta t\, \tau(t_{n-1})\right) - \Delta t\, \tau(t_n)$$

$$= A^2 e_{n-1} - \Delta t A\, \tau(t_{n-1}) - \Delta t\, \tau(t_n)$$

$$e_{n-1} = A e_{n-2} - \Delta t\, \tau(t_{n-2})$$

$$\boxed{e(t_{n+1}) = A^{n+1} e(t_0) - \Delta t \sum_{i=0}^{n} A^i \tau(t_{n-i})}$$

$n+1 \to n$

$$e(t_n) = A^n e(t_0) - \Delta t \sum_{i=0}^{n-1} A^i \tau(t_{n-1-i})$$

$e(t_0) = 0$

because we start with IC

$d_0 = d(t_0)$

$$e(t_n) = -\Delta t \sum_{i=0}^{n-1} A^i \, \underbrace{\tau(t_{n-1-i})}_{\text{consistency}}$$

$A^i \rightarrow$ stability

$$|e(t_n)| = \left| -\Delta t \sum_{i=0}^{n-1} A^i \, |\tau(t_{n-1-i})| \right|$$

$|a+b| \leq |a| + |b|$

$$\leq \Delta t \sum_{i=0}^{n-1} |A|^i \, \underbrace{|\tau(t_{n-1-i})|}_{\leq C(\Delta t)^k}$$

$|A| \leq 1 \Rightarrow |A|^i \leq 1$ stability

$$\leq \Delta t \left\{ \sum_{i=0}^{n-1} (1) \quad C(\Delta t)^k \right\}$$

$$= \underbrace{\Delta t (n)}\, C(\Delta t)^k$$

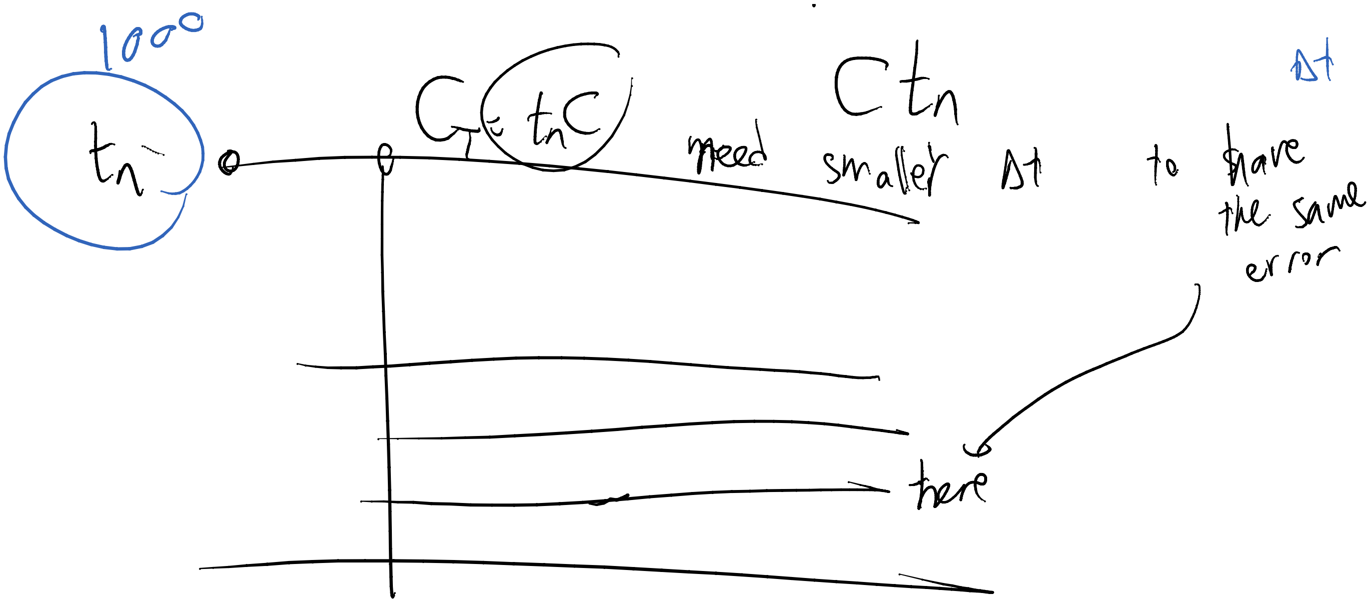
$$C t_n \, (\Delta t)^k$$

$$|e(t_n)| \leq C_T (\Delta t)^k$$

$k \rightarrow$ order of accuracy "convergence rate"

$C_T \rightarrow$ depends on time $\rightarrow$ problem would have been if C_T was dependent on Δt

1000



λ repeated 4 times $n_\lambda^A = 4$

2 eigenvector independent $n_\lambda^G = 2$