2016/04/04

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11:38 AM

Solve eigen values of a 5x5 matrix

$\lambda_1 = 2$ $\lambda_2 = 3$

$n_1^A = 4$, $m_1 = 4$ $n_2^A = n_2 = 1$ $\# \eta s = n_1^A - n_1^G = 2$

multiplicity

$n_1^G = 2$ $n_2^G = 1$

$n_1^A - n_1^G = 2$

$$J = \begin{bmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & \cancel{1} & \\ & & & 2 & \\ & & & & 3 \end{bmatrix}$$

$n_1^G = 3$

$n_1^A - n_1^G = 1$

$Au_1 = \lambda_1 u_1$

what is the dimension of $\{u_1 | Au_1 = \lambda u_1\}$

$$\begin{bmatrix} 2 & 1 & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 3 \end{bmatrix}$$

$n_1^G = 4$

$= n_1^A$

$$\begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 3 \end{bmatrix}$$

Spectral stability:

${}^{t+n\Delta t}\hat{\mathbf{X}} = \mathbf{A}^{nt}\hat{\mathbf{X}}$ is stable iff $\rho(\mathbf{A}) \leq 1$ and if $\mathbf{A}$ is **not** diagonalizable eigenvalues a_i with $n_i^A > n_i^G$ satisfy $\lvert a_i \rvert < 1$

- $\rho(A) \leq 1$ if A is diagonalizable

If not diagonalizable some root have the condition

eigenvalues a_i with $n_i^A > n_i^G$ satisfy $\lvert a_i \rvert < 1$

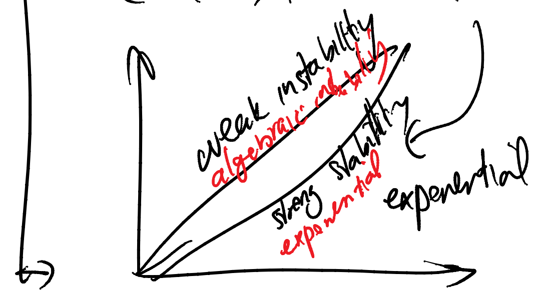
for these reals $|a_i|<1$

$$\mathbf{J} = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \quad \Rightarrow \mathbf{J}^n = \begin{bmatrix} a^n & na^{n-1} \\ 0 & a^n \end{bmatrix}$$

$|a| < 1$ $\quad n a^{n-1} \to 0$

$|a| = 1$ $\quad |na^{n-1}| = n$

$|a| > 1$ $\quad$ blows up a^n $\quad |a|>1$



Using spectral analysis analyze stability of a few methods

Stability analysis of LMS methods 411

Linear Multistep

general $\quad$ linear

$$\dot{\mathbf{y}} = f(\mathbf{y}, t) = \mathbf{G}_0\mathbf{y} + \mathbf{H}(t)$$
$$\ddot{\mathbf{y}} = f(\mathbf{y}, \dot{\mathbf{y}}, t) = \mathbf{G}_1\dot{\mathbf{y}} + \mathbf{G}_0\mathbf{y} + \mathbf{H}(t)$$

$$\sum_{i=0}^{k} \{\alpha_i \mathbf{y}_{n+1-i} + \Delta t \beta_i \underbrace{[\mathbf{G}_0 \mathbf{y}_{n+1-i} + \mathbf{H}(t_{n+1-i})]}_{f(y_{n+1-i},\, t)}\} = 0 \quad \text{LMS applied Linear first order ODE}$$

$$\dot{y}_n \approx \frac{y_{n+1} - y_n}{\Delta t}$$

$$\dot{y}_n = f(y_n, t)$$

$$\frac{y_{n+1} - y_n}{\Delta t} = f(y_n, t)$$

$$y_{n+1} = y_n + \Delta t f(y_n, t)$$

k steps prior to y_{n+1}

y_{n+1} ∘

y_n ∘

y_{n-1} ∘

y_{n+1-k} ∘

$\alpha y_{n+1} + \alpha \Delta t f(y_n, t)$ are used to update y_{n+1}

$+(1)\, y_n + (1) \Delta t\, f(y_n, t) = 0$

$$\dot{y}_n = \frac{y_{n+1} - y_{n-1}}{2\Delta t} \Rightarrow \text{different } \alpha\text{'s \& } \beta\text{'s}$$

first order

$$\sum_{i=0}^{k}\{\alpha_i \mathbf{y}_{n+1-i} + \Delta t\beta_i[\mathbf{G}_0\mathbf{y}_{n+1-i} + \mathbf{H}(t_{n+1-i})]\} = 0 \quad \text{LMS applied Linear first order ODE} \quad (341a)$$

$$\sum_{i=0}^{k}\{\alpha_i \mathbf{y}_{n+1-i} + \Delta t\beta_i\mathbf{G}_1\mathbf{y}_{n+1-i} + \Delta t^2\gamma_i[\mathbf{G}_0\mathbf{y}_{n+1-i} + \mathbf{H}(t_{n+1-i})]\} = 0 \quad \text{LMS applied Linear second order ODE} \quad (341b)$$

2nd order

$$\frac{y_{n+1}}{\begin{Bmatrix} y_n \\ y_{n-1} \\ \vdots \\ y_{n-k+1} \end{Bmatrix}}$$

this is a recursive equation writing y_{n+1} in terms of k previous steps $n, \ldots, n-k+1$.

$\dot{x} = -\lambda x + f(t)$
$= G_0 x + H(t)$

MDOF	$\Rightarrow$	SDOF	Parameters in (340)	
$\mathbf{M}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$	$\Rightarrow$	$\dot{x} + \lambda x = f(t)$	$\mathbf{G}_0 = -\lambda, \mathbf{H}(t) = f(t)$	(342a)
$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$	$\Rightarrow$	$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f(t)$	$\mathbf{G}_1 = -2\xi\omega, \mathbf{G}_0 = -\omega^2, \mathbf{H}(t) = f(t)$	(342b)

$$\ddot{x} = \underbrace{-2\xi\omega}\, \dot{x} - \omega^2 x + f(t)$$

$$\ddot{x} = G_1 \dot{x} + G_0 x + H(t)$$

$G_0 x + H$

SDOF

$$\sum_{i=0}^{k}\{\alpha_i x_{n+1-i} + \Delta t\beta_i[-\lambda x_{n+1-i} + f(t_{n+1-i})]\} = 0 \qquad \text{LMS for } \dot{x} + \lambda x = f(t) \qquad (343a)$$

$$\sum_{i=0}^{k}\{\alpha_i x_{n+1-i} - 2\Delta t\beta_i\xi\omega x_{n+1-i} + \Delta t^2\gamma_i[-\omega^2 x_{n+1-i} + f(t_{n+1-i})]\} = 0 \qquad \text{LMS for } \ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f(t) \qquad (343b)$$

Let $f=0$

x_{n+1}

x_n

x_{n-k+1}

k steps

$$\sum_{i=0}^{n} \alpha_i x_{n+1-i} + \Delta t \beta_i (-\lambda x_{n+1-i}) = 0$$

$$\sum \alpha_i x_{n+1} - 2\Delta t \beta_i \xi \omega x_{n+1-i} + \Delta t^2 \gamma_i (-\omega^2) x_{n+1-i} = 0$$

$$\sum_{i=0}^{n} \underbrace{\left(\alpha_i - \lambda \Delta t \beta_i\right)}_{c_i} x_{n+1-i} = 0$$

they depend on Δt & $\alpha_i, \beta_i, \gamma_i$ which paramters of the multi step methods

$$c_0 x_{n+1} + c_1 x_n + c_1 x_{n-1} + \cdots + c_k x_{n-k+1} = 0 \quad \text{where } c_i = \begin{cases} \alpha_i - \Delta t \beta_i \lambda & \text{LMS for } \dot{x} + \lambda x = 0 \\ \alpha_i - 2\Delta t \beta_i \xi \omega - \Delta t^2 \gamma_i \omega^2 & \text{LMS for } \ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = 0 \end{cases} \tag{344}$$

$$x_{n+1} = \bar{c}_1 x_n + \bar{c}_1 x_{n-1} + \cdots + \bar{c}_k x_{n-k+1}, \quad \bar{c}_i = -\frac{c_i}{c_0}$$

$x_0 = 1$

$x_1 = 1$

$x_{n+1} = 1 x_n + 1 x_{n-1}$

1
1
2
3
5
8

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amplification scalar

$x_{n+1} = A x_n$ if this was true $\longrightarrow$ $x_n = A^n x_0$

$|A| \leq 1$ for stability

$$c_0 x_{n+1} + c_1 x_n + c_1 x_{n-1} + \cdots + c_k x_{n-k+1} = 0$$

$$\left(c_0 A^k + c_1 A^{k-1} + c_k\right) x_{n-k+1} = 0$$

$$x_{n+1} = A^2 x_{n-1} = A^3 x_{n-2} = A^k x_{n-k+1}$$

$$x_{n+1} = A^2 x_{n-2} = A^3 x_{n-3} = A^k x_{n-k+1}$$

$$\boxed{c_0 A^k + c_1 A^{k-1} + \dots + c_k = 0}$$

k'th order polynomial

roots should be ≤ 1

$|A_i| \le 1$

if repeated $\rightarrow$ $|A_i| < 1$

$$\hat{\mathbf{X}}_n = \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-k+1} \end{bmatrix}$$

update vector

$$\begin{bmatrix} x_{n+1} \\ x_n \\ | \\ x_{n-k+2} \end{bmatrix} = \begin{bmatrix} \bar{c}_1 & \cdots & \bar{c}_k \\ 1 & 0 & \\ & 1 & 0 \\ & & 1 \quad 0 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-k+1} \end{bmatrix}$$

$$x_{n+1} = \bar{c}_1 x_n + \bar{c}_1 x_{n-1} + \cdots + \bar{c}_k x_{n-k+1},$$

$x_n = (1) x_n$

$k = 2$ $\begin{bmatrix} \bar{c}_1 & \bar{c}_2 \\ 1 & 0 \end{bmatrix}$

$k = 3$ $\begin{bmatrix} c_1 & \bar{c}_2 & c_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$\hat{X}_{n+1} = \overset{\text{matrix}}{A} \hat{X}_n$$

$$\hat{\mathbf{X}}_n = \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \end{bmatrix}$$

$$X_{n+1} = \underbrace{\quad}_{k \times k} X_n \qquad \hat{\mathbf{X}}_n = \begin{bmatrix} x_{n-1} \\ \vdots \\ x_{n-k+1} \end{bmatrix}$$

eigen values $|A_i| \leq 1$

If A_i is repeated & $n_i^A > n_i^G$ $|A_i| < 1$

$$\begin{bmatrix} \bar{c}_1 & \bar{c}_2 & \bar{c}_3 & \cdots & \bar{c}_k \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_k \end{bmatrix} = \underset{\text{eigenvalue}}{a} \underset{\text{eigenvector}}{\begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_k \end{bmatrix}}$$

$$\bar{c}_1 v_1 + \bar{c}_2 v_2 + \cdots + \bar{c}_k v_k = a v_1$$

$$v_1 = a v_2$$

$$v_2 = a v_3 \implies v_i = a^{k-i} v_k$$

$$\vdots$$

$$v_{k-1} = a v_k$$

$$\left(\bar{c}_1 a^{k-1} + \bar{c}_2 a^{k-2} + \cdots + \bar{c}_k\right) v_k = a^k v_k$$

$$a^k - \bar{c}_1 a^{k-1} - \bar{c}_2 a^{k-2} - \cdots - \bar{c}_k = 0$$

$$\bar{c}_i = -\frac{c_i}{c_0}$$

$$\boxed{a_0 a^k + c_1 a^{k-1} \; --- \; + c_k = 0}$$

for this matrix all eigenvalues only have one eigenvector

(because of $U^i = a^{i-k} U_k$), i.e $n_i^G = 1$

so if a root a_i is repeated it has the condition $n_i^A > 1 = n_i^G$
$n_i^A > 1$

$\Longrightarrow$ $|a_i| < 1$

& cannot be 1

So roots of $a_0 a^k \; --- \; + a_k = 0$

$|a_i| \leq 1$ if simple

$|a_i| < 1$ " repeated

Example for central difference & Houbolt methods are given in the course notes