2016/04/06

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$$a_p \frac{\mathrm{d}^p u}{\mathrm{d}t^p} + a_{p-1} \frac{\mathrm{d}^{p-1} u}{\mathrm{d}t^{p-1}} + \cdots + a_1 \frac{\mathrm{d}u}{\mathrm{d}t} + a_0 u(t) = 0 \quad \Rightarrow \quad u(t) = \sum_{i=1}^{\bar{p}} P_i(t) e^{\lambda_i t} \tag{368}$$

Numerical solution of systems as equation (368)

5.3.2.2 Absolute stability

- Consider the first order ODE,

$$\dot{x} - \lambda x = 0 \tag{373}$$

How can we solve (and also analyze) the stability of a p-th order ODE with the solution and analysis of first order ODE (373):

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = f(t)$$

$d = x$

$v = \dot{x}$

$$\dot{v} + 2\xi\omega v + \omega^2 d = f(t)$$

$$\dot{d} - v = 0$$

$$\underbrace{\begin{bmatrix} \dot{d} \\ \dot{v} \end{bmatrix}}_{\dot{q}} + \underbrace{\begin{bmatrix} 0 & -1 \\ \omega^2 & 2\xi\omega \end{bmatrix}}_{\Lambda} \begin{bmatrix} d \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ f(t) \end{bmatrix}}_{F}$$

$$\dot{q} + \Lambda q = F(t)$$

example generalized alpha

you can solve this with any first order ODE solver such as RK, generalized alpha, etc.

$$\dot{q}^{n+\alpha} + \Lambda q^{n+\alpha} = F(t)_{n+\alpha}$$

$$\dot{q}^{n+\alpha} = \frac{q^{n+1} - q^{n}}{\Delta t} \qquad q^{n+\alpha} = \left[(1-\alpha) q_n + \alpha q_{n+1}\right]$$

$$\left[I_{2\times 2} + (1-\alpha)\Lambda\right] q_{n+1} = \left[I_{2\times 2} - \alpha\Lambda\right] q_n + \Delta t f(t_{n+\alpha})$$

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = 0$$

$$e^{\lambda t} \qquad \underline{\lambda^2 + 2\zeta\omega\lambda + \omega^2 = 0}$$

$$\boxed{\lambda_{1,2} = -\zeta\omega \pm \sqrt{\zeta^2\omega^2 - \omega^2}}$$

$$e^{\lambda_1 t} \qquad e^{\lambda_2 t}$$

if $\zeta = 0$ $\qquad \underline{\lambda_{1,2} = \pm\omega i}$ $\qquad u = Ae^{\omega i} + Be^{-\omega i}$

$$= \acute{A}\cos\omega t + \acute{B}\sin\omega t$$

$$\ddot{x} + \omega^2 x = 0$$

$\zeta < 1$ $\qquad \boxed{\lambda_{1,2} = -\zeta\omega \pm i\omega\sqrt{1-\zeta^2}}$

$$\lambda \qquad \lambda_T$$

$\xi < 1$ $\boxed{\lambda_{1,2} = -\xi\omega \mp i\omega\sqrt{1-\xi^2}}$

λ_R λ_I

$$A_1 e^{(\lambda_R^1 + i\lambda_I^1)t} + A_2 e^{(\lambda_R^2 + i\lambda_I^2)t}$$

$$u(t) = A_1 e^{\lambda_R^1 t} e^{i\lambda_I^1 t} + A_2 e^{\lambda_R^2 t} e^{i\lambda_I^2 t}$$

≤ 0 ≤ 0

if $\lambda_1 = \lambda_2$; $\xi = 1$ $\lambda_{1,2} = -\omega$

$$u(t) = A_1 e^{-\omega t} + A_2 t e^{-\omega t}$$

Why the stability analysis of

$$a_p \frac{\mathrm{d}^p u}{\mathrm{d}t^p} + a_{p-1}\frac{\mathrm{d}^{p-1} u}{\mathrm{d}t^{p-1}} + \cdots + a_1 \frac{\mathrm{d}u}{\mathrm{d}t} + a_0 u(t) = 0 \quad (*)$$

we only consider ODE's where all $\lambda_i^R < 0$

reduces to stability analysis of

$$\dot{u} - \lambda_i u = 0$$

λ_i is a root of $(*)$

Real solution

$$u = P_i(t) e^{\lambda_i t}$$

real $u = \underbrace{P_i(t)\, e^{\lambda_i t}}$

this is the solution

to $\dot{u} - \lambda_i u = 0$

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we are interested in evaluating if a numerical method can stably solve

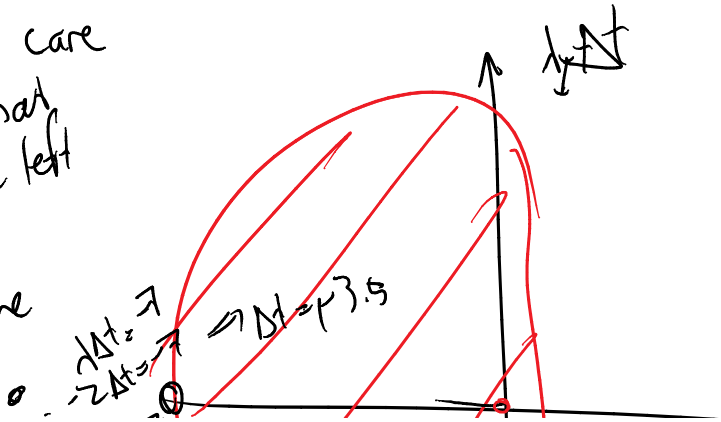
$$\dot{u} - \lambda u = 0$$

for $\lambda = \underline{\lambda_R + i\,\lambda_I \qquad \lambda_R < 0}$

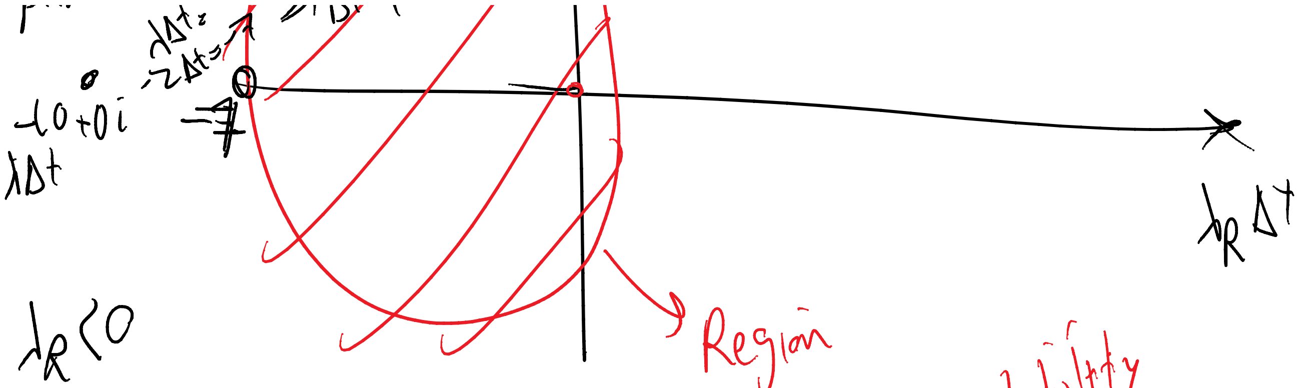
problems where physical solution is diminishing

time marching scheme has a Δt

we care about the left half plane



$e^{\lambda_R t}\, e^{i \lambda_I t}$



Region of absolute stability of a time march method is where the solution to

$\dot{u} + 2u = 0$

$\lambda = -2 + 0i$

$\Delta t = 5$

$\lambda\Delta t = -10 + 0i$

$\dot{u} - \lambda u = 0$ is stable & tending to zero as $t \to \infty$

for time step Δt

Example Generalized Alpha

Real

$\dot{u} - \lambda u = 0$ λ real

$$(1 - \alpha\Delta t\ \lambda^h)d_{n+1} = (1 + (1-\alpha)\Delta t\ \lambda^h)d_n$$

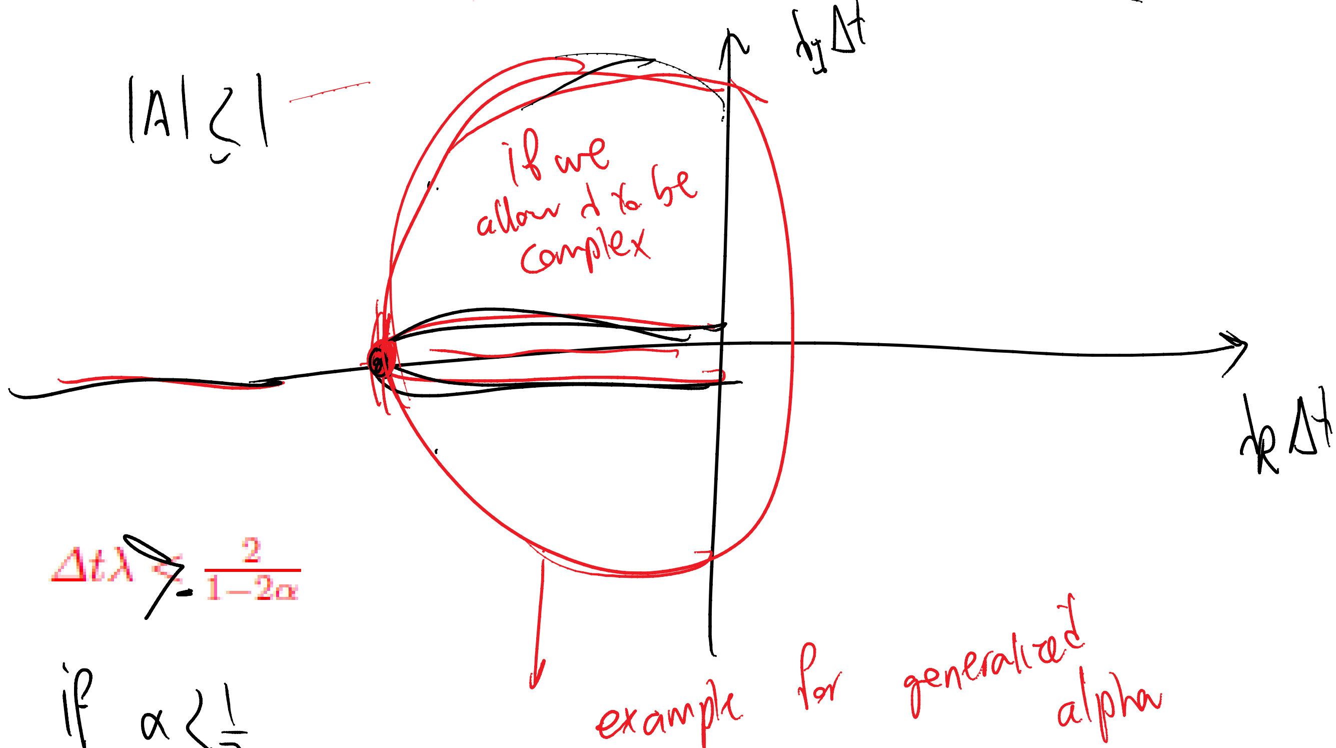
$$d_{n+1} = Ad_n, \quad \text{where} \quad A = \frac{1 + (1-\alpha)\Delta t\lambda}{1 - \alpha\Delta t\lambda} \quad \text{Amplification factor}$$

$\alpha < \frac{1}{2}$

$\lambda_R \Delta t$

$d_{n+1} = A d_n$, where $A = \frac{\cdots}{1 + \alpha \Delta t \lambda}$ Amplification factor

$|A| \leq 1$



$\Delta t \lambda \geq \frac{2}{1-2\alpha}$

if $\alpha < \frac{1}{2}$

A - stable method

Absolutely stable method

it's a numerical method

that the region of absolute stability covers

the entire negative real half plane

This numerical method provides

stable solution for any Δt when

stable solution for any Δt when the underlying physical solution is dynamically stable.

$\dot{u} - \lambda u = 0 \qquad \lambda = \lambda_R + \lambda_I i$

$\lambda_R < 0$

numerical method is stable for $\underline{\text{Any}}$ Δt

The difference between the concepts of A-stability and unconditional stability

$$\dot{x} - \lambda_0 x = 0, \quad (\lambda_0^I = 0, \lambda_0^R < 0) \quad \star$$

fixed

Assume λ_0 is real & negative

$$x = A e^{\lambda_0 t}$$

A numerical method for $\star$ is unconditionally stable if we can take arbitrary time step while still the numerical method is stable

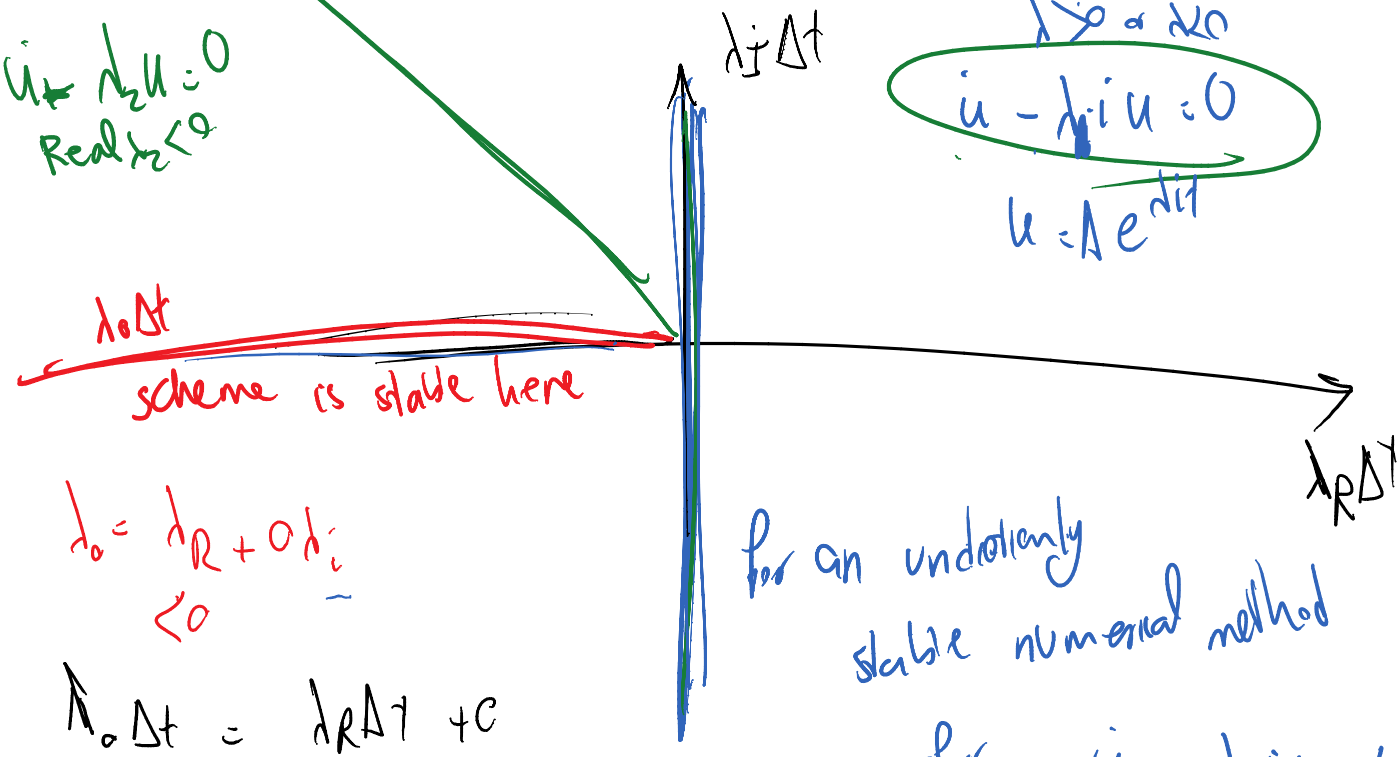
$\dot{u} + \lambda_2 u = 0$

Real $\lambda_2 < 0$

$\lambda > 0$ or $\lambda < 0$

$\dot{u} - \lambda_I i u = 0$

$u = A e^{\lambda i t}$



$\lambda_0 = \lambda_R + 0 \lambda_i$

< 0

$\lambda_0 \Delta t = \lambda_R \Delta t + 0$

let Δt go from 0 to ∞

For an unconditionally stable numerical method for $\dot{u} - \lambda_I i u = 0$

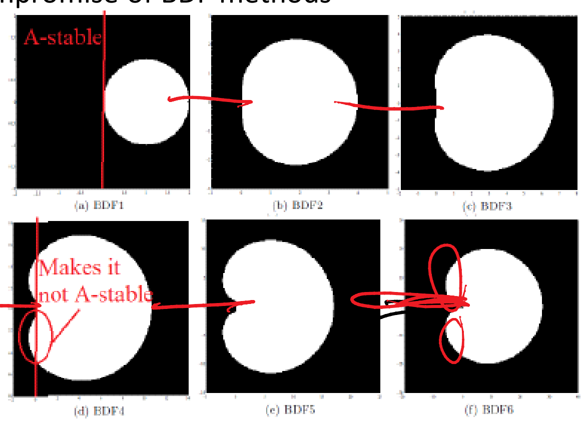
λ_I is fixed

Can we get "well-behaved" numerical methods LMS

that are higher order than 2

with some compromise on A-stability

Compromise of BDF methods



Absolute stability regions: the region of absolute stability is the set of points in the complex plane outside the white region

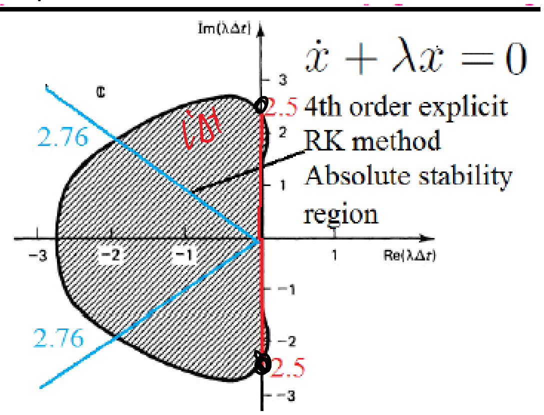
The region of absolute stability contains all the negative real axis.

these methods are unconditionally stable for solving

$$\dot u - \lambda u = 0$$

$$\lambda = \lambda_R + i\lambda_i, \quad \lambda_R < 0, \quad \lambda_i \gtrless 0$$

Use in practice

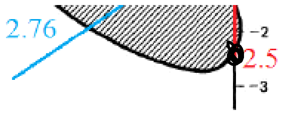


$$\ddot x + x = 0$$

$$x = e^{\lambda t} \qquad \lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$x = A_1 e^{it} + A_2 e^{-it}$$



$$x = A_1 e^{it} + A_2 e^{-it}$$

$$\lambda_{1,2} = \pm i$$

$$\lambda_1 \Delta t = i\,\Delta t \qquad |i|\,\Delta t < 2.5$$
$$\lambda_2 \Delta t = -i\,\Delta t \qquad |-i|\,\Delta t < 2.5$$
$$\Rightarrow \underline{\Delta t < 2.5}$$

why $\ddot{x} + x = 0$ corresponds to hyperbolic PDE

$$\ddot{u} - c^2 u_{,xx} = 0 \Longrightarrow \underset{\text{MDOF}}{M\ddot{U} + KU = 0}$$

Modal analysis $\Longrightarrow$ $\ddot{x}_i + \omega_i^2 x = 0$

$$\ddot{x} + \underbrace{2\dot{x}}_{2\xi\omega} + \underbrace{2x}_{\omega^2} = 0 \qquad \Rightarrow \lambda^2 + 2\lambda + 2 = 0$$

$$\Rightarrow \lambda = -1 \pm \imath \qquad \Rightarrow x = a_1 e^{(\imath - 1)t} + a_1 e^{(-\imath - 1)t}$$

$$M\ddot{U} + C\dot{U} + KU = 0$$

Assuming we can go to SDOF $\Longrightarrow$ e.g. $C = \alpha M + \beta K$

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = 0$$

$$\lambda = \overset{\xi\omega}{(-1)} \pm \overset{\omega_d}{\imath} \qquad \Rightarrow x = a_1 e^{(\imath - 1)t} + a_1 e^{(-\imath - 1)t}$$

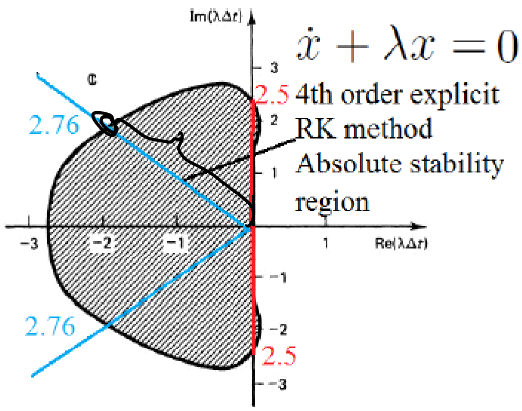
$$\lambda = -1 \pm i \Rightarrow x = a_1 e^{(i-1)t} + a_1 e^{(-i-1)t}$$

oscillatory

diminishing

$\lambda_R \leq 0$ great physical solution diminishing

$\lambda = \underline{-1} \pm i$ $|\lambda| = \sqrt{2}$



$2.76 = |\lambda| \Delta t$

$= \sqrt{2} \Delta t$

$\Rightarrow$ $\boxed{\Delta t \leq 1.94}$

5.4.1 Control of high frequency numerical noise

$\Delta t \leq \frac{2}{\omega} = \frac{2}{2\pi} T = \frac{1}{\pi} T$

- In the figure observe spectral radius of different time marching methods versus normalized element size.
- $T = \frac{\omega}{2\pi}$ is the period of a given SDOF.
- Clearly, as expected central-difference method becomes unstable for $\Delta t/T > \frac{1}{\pi}$: As we observed in (357) (also (358)) central difference method is stable if $\Delta t\omega \leq 2$, $T = \frac{\omega}{2\pi} \Rightarrow \Delta t/T \leq \frac{1}{\pi}$
- Other methods in the figure are unconditionally stable.
- One very important aspect of a time marching method in these plots is,

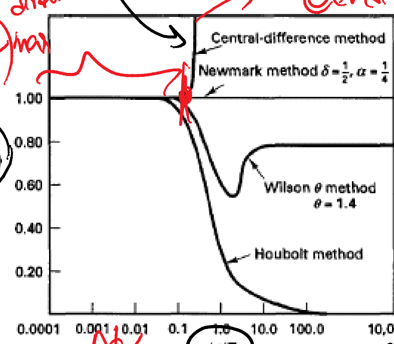
$$\rho_\infty = \lim_{\Delta t/T \to \infty} \rho(\mathbf{A}(\frac{\Delta t}{T})) \qquad (387)$$

for example for Wilson-θ method $\rho_\infty \approx 0.8$

for control difference

$(\frac{\Delta t}{T})_{max}$

conditionally stable



$\Delta t / T$

spectral radius of Amplification factor

$\frac{\Delta t}{T}$ period

$u^{n+1} = A u^n$

$\rho(A)$

spectral radius

T period =

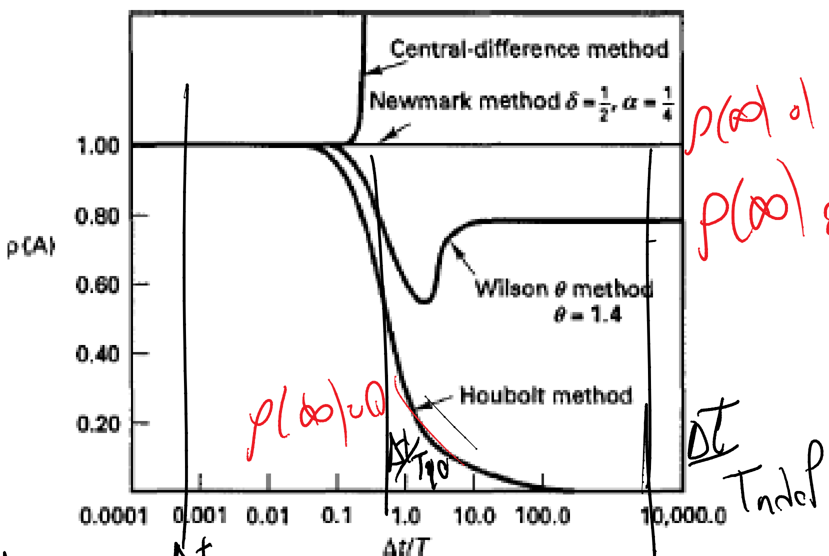
$2\pi = T\omega$

$\boxed{T = \frac{2\pi}{\omega}}$

ODE of consideration

$\ddot{x} + \omega^2 x = 0$

$\boxed{T = \frac{2\pi}{\omega}}$



$\frac{\Delta t}{T_1}$

$\ddot{x} + \omega^2 x = 0$

$\omega = \frac{2\pi}{T}$

$\frac{\Delta t}{T}$

MDoF

$M\ddot{U} + KU = 0$

$\ddot{x} + \omega_i^2 x = 0$

$\omega_1 < \omega_2 \ldots \omega_{ndof}$

very large

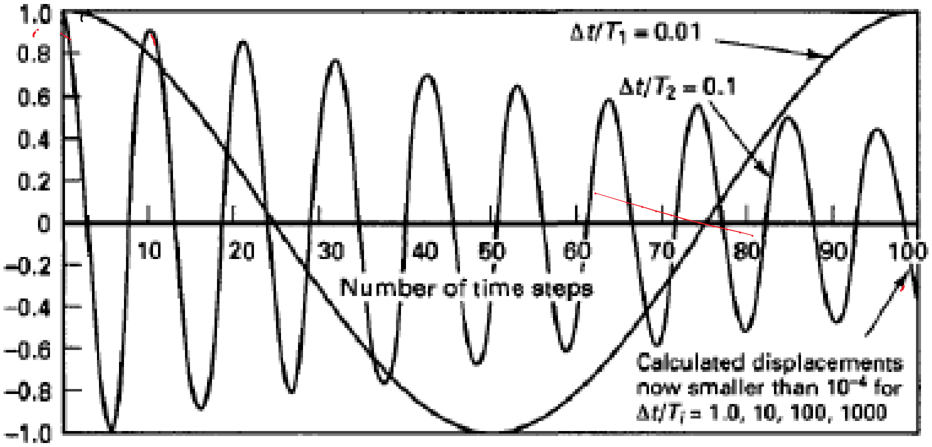
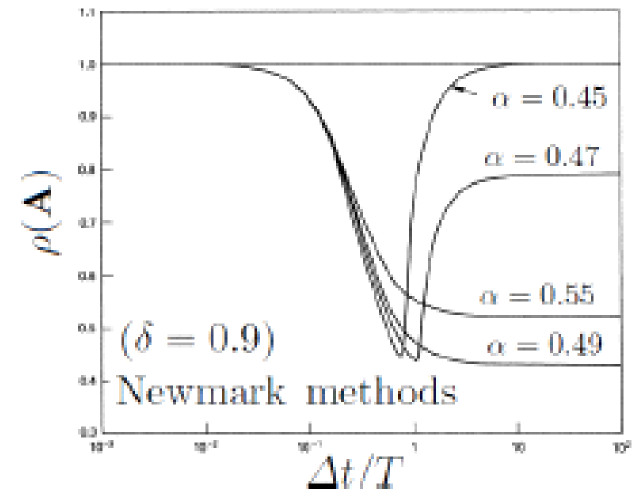


Figure 9.7 Displacement response predicted with increasing $\Delta t/T$ ratio; ~~Wilson~~ θ method, $\theta = 1.4$ [Bathe, 2006]



$$\alpha \approx \frac{\left(\delta+\frac{1}{2}\right)^2}{4}:$$

best
for Newmark
method